Stability and Minimum Cost Analysis of a Discrete-Time Disturbance Accommodation Controller

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STABILITY AND MINIMUM COST ANALYSIS OF A DISCRETE-TIME DISTURBANCE ACCOMMODATION CONTROLLER

by

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ABSTRACT

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Disturbance accommodation control (DAC) is a method for designing a controller that minimizes the effects of disturbances of known waveform type, but with unknown arrival time, duration or magnitude. Systems that do not have a control term in the measurement equation pose a particular challenge for DAC design. A disturbance accommodation controller for these types of systems was previously developed by defining a pseudo-output consisting of the current output and previous control input terms with weighting coefficients.

The objective of the present work is to analyze the stability and performance of the discrete- time disturbance accommodation controller for systems without a feed forward term in the measurement equation. Three example systems each of first, second, and third order are used in this analysis. As a result of extensive graphical analyses, recommendations are made to enable designers to set appropriate limits on the range of the controller coefficients to ensure closed loop system stability and disturbance attenuation for a minimum cost. In addition, guidance is given for making an appropriate choice of sampling time to discretize continuous time systems to ensure closed loop system stability when using this type of DAC controller.

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Chapter 1: Introduction

Control systems are key for the successful design, implementation, and operation of countless engineering systems such as vehicles, machines, and robots, along with many others. Having such different possible applications for control systems leads for the need for many different types of controllers. One such type of controller that will be discussed and investigated in this work is the Disturbance Accommodation Controller.

1.1: Review of Disturbance Accommodation Control

Disturbance Accommodation Control (DAC) is a method for designing controllers that minimizes the effects of disturbances on a given system. To design a DAC, the system must be able to be represented as a state space model with a disturbance signal of known waveform type, such as a step, ramp, sinusoidal, or exponential, but unknown arrival time, duration and magnitude. Disturbance accommodation control is a very useful technique for many practical applications. One such application of disturbance accommodation control is related to speed regulation and mitigation of drive-train torsion fatigue in flexible wind turbines [1]. In this work, particular disturbance accommodation control techniques called disturbance utilization control is used, which attempts to use the energy in the disturbance signal to help achieve the control objective. Using the energy in turbulent wind inflow to design a disturbance accommodation controller decreased the demand on the actuators as well as dampened the torsion oscillations of the drive-train of the wind turbine.

1.2: Literature Review

Disturbance Accommodation Control (DAC) has been developed and used in many applications to reduce the effect of disturbances applied to various systems. A main contributor to DAC theory is C.D. Johnson, who has developed DAC techniques for applications related to linear regulator and servomechanism problems [2]. His techniques took the effects of fluctuating external disturbances into account when designing the controller. Johnson's techniques for designing the controllers make use of the disturbances in the system to help accomplish the main controller goals. It is pointed out in this work that there are three possible objectives when designing disturbance accommodation controllers. Firstly, it can be assumed that the effect of the disturbances on the system is undesirable and that the controller must completely counteract the disturbances. Secondly, when the effects of the disturbances cannot be eliminated completely, the controller could be designed to minimize the effect of the disturbances. Lastly, the disturbances acting on the system could be beneficial towards accomplishing the primary control objective. These three types of disturbance accommodation control are respectively referred to as complete disturbance cancellation (rejection), disturbance minimization, and optimal disturbance utilization [1]. In the work proposed in this thesis, disturbance minimization techniques will be used.

C.D. Johnson also extended his DAC applications to stabilization problems, set-point regulation, and servo-tracking control problems. [3] The various methods for designing disturbance accommodation controllers previously mentioned were expanded upon, including a Disturbance-Absorption Mode of Accommodation, a Disturbance-Minimization Mode of Accommodation, a Disturbance-Utilization Mode of Accommodation and a Multi-Mode Accommodation of Disturbances.

These techniques for minimizing the effects of disturbances with known waveforms were expanded upon in [4], applying DAC techniques to discrete-time non-linear stochastic systems. This approach modeled the applied disturbance of known waveform type as the output to a linear system driven by impulses. The state of the disturbance was then estimated from the input/output data to combine with a least squares disturbance accommodation controller. Disturbance accommodation for nonlinear systems having nonlinear disturbance models was also investigated [5]. This control technique involves representing the disturbance as a linear model and using the composite nonlinear system to develop a controller that dynamically minimizes the disturbance.

To further the design of this type of controller, a disturbance accommodation controller using Linear Matrix Inequality (LMI) techniques for systems with multiple state and input delays was designed [6]. This LMI based design relies on the use of an observer with multiple time delays to minimize the effect of the disturbance signal on the system. On another occasion, this method of disturbance accommodation control was revisited through the use of an LMI based reduced order observer design [7]. Using a reduced-order observer based design while still achieving disturbance accommodation control as well as the primary control objectives showed that the use of a full order observer is not always necessary.

State estimation and regulation was investigated when low frequency disturbances and uncertainties are present [8]. This design consisted of a Linear Quadratic Gaussian (LQG) type controller to accommodate the effects of Gaussian disturbances. Since this controller becomes sub-optimal when non-Gaussian disturbances are applied, the controller was modified to eliminate the non-Gaussian disturbances by using disturbance cancellation and filters. Although using a different controller, somewhat similar techniques were used in the design described in this thesis.

Lastly, recently a time-optimal deadbeat DAC using first a full-order observer was designed followed by a reduced-order observer based design [9]. These designs considered a discrete-time system in canonical form, with a disturbance of known waveform present in both the states of the system and the measurement. While stability was guaranteed and disturbance accommodation was achieved, a large control magnitude was required proving impractical for implementation. Continuing the investigations of this controller, the DAC was re-designed to minimize the magnitude of the control input [10]. As for the previous controller, discrete systems with feed forward control terms present in the measurement of the system were used. While still time optimal, the newly designed controller was no longer deadbeat therefore stability was not guaranteed.

1.3: Contribution

As previously described, several disturbance accommodation controllers have been designed for application to various linear and non-linear system models as well as disturbance models. The objective of the present work is to analyze the stability and performance of a Disturbance Accommodation Controller applied to discrete-time systems without a feed forward control term in the measurement equation. The lack of a control term in the measurement equation of most practical systems proves problematic when applying any of the aforementioned controllers. Therefore, to introduce a factor of the control term into the measurement in an indirect manner, the concept of a "pseudo-output" equation is considered. This pseudo-output consists of the current output and previous control input terms. Weighting coefficients associated with the pseudo-output provide additional freedom when designing the controller.

The main controller objectives investigated in the design and analysis of this DAC are stability, disturbance attenuation and the minimization of the control input. Unlike previous contributions to disturbance accommodation control techniques, [9] this controller is not deadbeat and therefore stability cannot be guaranteed. To ensure stability of the system, the limits of the weighting coefficients of the pseudo-output equation will be analyzed. The effect of a disturbance of known waveform, applied to the state of the system as well as the measurement will be reduced by the application of the DAC. The effect of the pseudo output's weighting parameters on the controller's ability to accommodate the disturbance will also be investigated. Taking the results obtained from the design of the deadbeat DAC in [9] into consideration, the controller and analysis in this work will also aim to minimize the control input, allowing for more practical implementation of the controller.

1.4: Thesis Organization

Now that the main objectives of this work have been presented, the proposed technique, case studies and considerations for implementation will follow in subsequent chapters. Following the first chapter, the Proposed Technique will be discussed in Chapter 2. The proposed technique for this work will consist of a System Development, Controller Design, and Analysis section. Upon the complete description of the proposed techniques, the case studies will be introduced in Chapter 3. To demonstrate the designed controller first order, second order, and third order systems will be considered. Each system will be analyzed according to the previously outlined technique, whereupon the trending results will be discussed. Following the analysis of the case studies, Chapter 4 will contain Considerations for Implementation. Considerations for Implementation will include discussions on how to implement a system in continuous-time versus discrete-time, as well as any other conditions on the system or controller that may result from such considerations. To conclude this thesis, a summary of the results obtained and future work will be discussed in Chapter 5.

Chapter 2: Proposed Technique

The techniques and primary objectives outlined in the previous chapter will be expanded upon in this chapter, discussing in detail the procedure for the design and analysis of the Disturbance Accommodation Controller (DAC). Before designing the controller, the system and disturbance models will be introduced in the system development, followed by the controller design. To complete the chapter, the proposed analysis techniques will be discussed.

2.1: System Development

The systems considered for the development of this controller are single-input single-output, discrete time systems in controllable canonical form. By using the controllable canonical form of the system, the controllability will be guaranteed and the mathematical development of the design will be simplified. For this work, only systems without an input term in the measurement equation are considered, where it is not possible to use the control action in the measurement equation to reduce the effect of the disturbance on the output. The system model is shown below.

$$\begin{aligned} x(k+1) &= \overline{A}x(k) + \overline{B}u(k) + \overline{F}w(k) \\ y(k) &= \overline{C}x(k) + \overline{G}w(k) \end{aligned} \tag{2.1}$$

In the model shown in (2.1), x(k) represents the state of the system, y(k) is the scalar output of the system, u(k) is the scalar control input and w(k) is the disturbance signal. The notation using a bar over each coefficient matrix represents the matrix in canonical form. As shown in the system model, both the state and the output of the system are affected by the disturbance signal w(k), where $\overline{F}$ and $\overline{G}$ represent the respective weighting coefficients. The respective coefficient matrices in canonical form for the state and measurement equations for an nth order system are shown, where g_0 is a scalar value

$$\overline{A} = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \\ -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-2} & -\alpha_{n-1} & -\alpha_n \end{bmatrix}, \overline{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ b_o \end{bmatrix}, \overline{F} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ b_o \end{bmatrix}$$

$$\overline{C} = [1 \quad 0 \quad \cdots \quad 0 \quad 0 \quad 0], \overline{G} = g_o \tag{2.2}$$

As previously mentioned, the disturbance must be of known waveform, but it can be of unknown magnitude, arrival time, and duration, in order for the DAC to successfully complete the control objectives. In general, the disturbance could be a step, ramp, or sinusoidal signal, where $\overline{E}$ represents the coefficient matrix in the disturbance model below.

$$w(k+1) = \overline{E}w(k) + \sigma(k) \tag{2.3}$$

The disturbance signal used in this thesis is a step-type signal, where $\overline{E} = 1$ in the given disturbance model. For this step-type disturbance signal, σ(k) represents a series of unknown impulses that may arrive or disappear in a random manner. Therefore, the step-type disturbance model used in this design is now presented below.

$$w(k+1) = w(k) + \sigma(k) \tag{2.4}$$

Figure 2-1 shows the unit step type disturbance signal used in all of the case studies and systems that are discussed in this work.

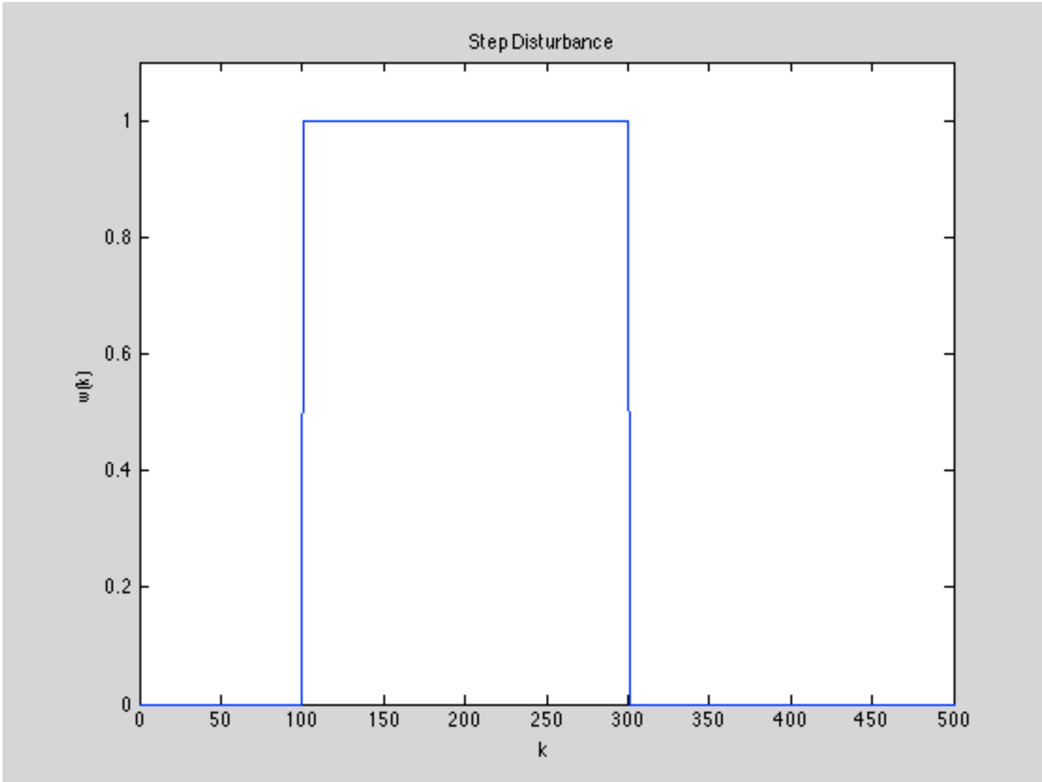

Figure 2- 1: Step-type disturbance signal

For values of the sampling instant k from 0 to 500, the step is being applied to the system from k=100 to k=300. The step is generated by using a value of σ(k) =1 for k=100, followed by σ(k) =-1 when k=300.

In order to compensate for the lack of input term in the output equation, the concept of a "pseudo-output" is introduced. The primary purpose of the pseudo-output is to indirectly introduce an input term in the output equation when it would normally not be present. The pseudo-output z(k) is composed of the sum of the current measurement along with the previous input term, with corresponding scalar weighting coefficients ϕ and γ [4].

$$z(k) = \phi y(k) + \gamma u(k-1) \tag{2.5}$$

The dynamic update equation for the pseudo output is:

$$z(k+1) = \left(\phi\overline{C}\overline{A}\right)x(k) + (\phi\overline{C}\overline{B} + \gamma)u(k) + \phi(\overline{C}\overline{F} + \overline{G}\overline{E})w(k) + \phi\overline{G}\sigma(k) \tag{2.6}$$

In this control design, the disturbance model in equation (2.3) consists of only the first term, because σ(k) represents the unknown series of impulses of the disturbance signal, when actually only its waveform-type is known. Unlike the previous control design in [9], the state update equation (2.1) is augmented with the pseudo-output update equation (2.6) instead of augmenting it with the output of the system. By including the pseudo-output in the composite system instead of the output of the system, the absence of the control input term in the output equation will be compensated for. This technique was explored in [3]. The resulting composite system is shown below.

$$\begin{bmatrix} x(k+1) \\ z(k+1) \end{bmatrix} = A_c \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} + B_c u(k) + F_c w(k) + G_c \sigma(k) \tag{2.7}$$

The state variables of the composite system consist of the original system state variables x(k) and the pseudo-output z(k). The coefficient matrices of the composite system are now in terms of the original system matrices as well as the weighting parameters of the pseudo-output, ϕ and γ.

$$A_c = \begin{bmatrix} \overline{A} & 0 \\ \phi\overline{C}\overline{A} & 0 \end{bmatrix},\ B_c = \begin{bmatrix} \overline{B} \\ \phi\overline{C}\overline{B} + \gamma \end{bmatrix},\ F_c = \begin{bmatrix} \overline{F} \\ \phi\left(\overline{C}\overline{B} + \overline{G}\overline{E}\right) \end{bmatrix},\ G_c = \begin{bmatrix} 0 \\ \phi\overline{G} \end{bmatrix} \tag{2.8}$$

Now that the composite system has been developed, the controller can be designed.

2.2: Controller Design

The control input is composed of a component $u_c(k)$ to control and stabilize the system, as well as a component $u_d(k)$ to accommodate the disturbance.

$$u(k) = u_c(k) + u_d(k) \tag{2.9}$$

When designing the controller state-feedback methods are used while assuming that the state variables are available for use. If the state variables are not available for use, then an observer can be designed to obtain estimates of the unknown state vector. The control input is composed of the controller component $u_c(k)$ and disturbance component $u_d(k)$.

$$u_c(k) = K_c x(k), \tag{2.10}$$

$$u_d(k) = K_d w(k) \tag{2.11}$$

The control and disturbance gains, K_c and K_d are chosen to minimize the state vector at the next iteration, as

$$\min_{K_c, K_d} \frac{1}{2} \left\| x_{k+1} \right\|^2 \tag{2.12}$$

where,

$$\begin{gathered} x_{k+1} = A_{CL} x_k + F_{CL} w_k \\ A_{CL} = A_c + B_c K_c, \; F_{CL} = F_c + B_c K_d \end{gathered} . \tag{2.13}$$

Expanding the square of the norm yields the following expression

$$\frac{1}{2}\left(x_{k+1}{}^T x_{k+1} \right) = \frac{1}{2}\left(x_k{}^T A_{CL}{}^T A_{CL} x_k + w_k{}^T F_{CL}{}^T F_{CL} w_k + 2 x_k{}^T A_{CL}{}^T F_{CL} w_k \right) \tag{2.14}$$

The cross-terms that result from the expansion of the square of the norm combine the effects of the state variables and disturbance signal, which complicate the minimization, so the upper bound on the norm is minimized. Instead, since

$$\left[2 x_k{}^T A_{CL}{}^T F_{CL} w_k \right] \le \left[x_k{}^T A_{CL}{}^T A_{CL} x_k + w_k{}^T F_{CL}{}^T F_{CL} w_k \right], \tag{2.15}$$

the norm can now be written in the following manner.

$$\min_{K_c,K_d} \frac{1}{2}\left\|x_{k+1}\right\|^2 \le \min_{K_c,K_d}\left[x_k{}^T A_{CL}{}^T A_{CL} x_k + w_k{}^T F_{CL}{}^T F_{CL} w_k\right] \tag{2.16}$$

Since x_k and w_k are independent variables, we have

$$\min_{K_c,K_d} \frac{1}{2}\left\|x_{k+1}\right\|^2 \le \min_{K_c,K_d}\left\|A_{CL} x_k\right\|^2 + \min_{K_c,K_d}\left\|F_{CL} w_k\right\|^2 \tag{2.17}$$

Performing a least-squares expansion on each term separately yields

$$\begin{aligned} \left\|A_{CL} x_k\right\|^2 &= x_k{}^T (A_c + B_c K_c)^T (A_c + B_c K_c) x_k \\ &= x_k{}^T \left(A_c{}^T A_c + A_c{}^T B_c K_c + K_c{}^T B_c{}^T A_c + K_c{}^T B_c{}^T B_c K_c\right) x_k \end{aligned} \tag{2.18}$$

By adding and subtracting a K_c^* term to equation (2.18) and therefore completing the square, a new norm-squared is defined in equation (2.19).

$$\left\|A_{CL} x_k\right\|^2 = x_k{}^T \left(A_c{}^T A_c + \left(K_c + K_c{}^*\right)^T B_c{}^T B_c \left(K_c + K_c{}^*\right) - K_c{}^{*T} B_c{}^T B_c K_c{}^*\right) x_k \tag{2.19}$$

The least-squares minimization is guaranteed as long as $K_c^* = - K_c$ in equation (2.19). By equating equations (2.18) and (2.19) and solving the expression for K_c^*, the controller gain K_c is determined.

$$\begin{aligned} K_c{}^T B_c{}^T B_c K_c{}^* &= K_c{}^T B_c{}^T A_c \\ K_c{}^* &= \left(B_c{}^T B_c\right)^{-1} B_c{}^T A_c \\ K_c &= -K_c{}^* = -\left(B_c{}^T B_c\right)^{-1} B_c{}^T A_c \end{aligned} \tag{2.20}$$

Considering the fact that F_{CL} is of the same form as A_{CL}, the same process is performed for the disturbance gain. Therefore, the resulting disturbance gain is

$$K_d = -(B_c{}^T B_c)^{-1} B_c{}^T F_c. \tag{2.21}$$

Now that the control and disturbance gains have been designed, they can be used in the control input u(k). Using the controller gains and disturbance gains defined above, the closed loop system representation is obtained by substituting the control input u(k) from(2.9) into the system model (2.7).

$$\begin{bmatrix} x(k+1) \\ z(k+1) \end{bmatrix} = A_{CL} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} + F_{CL} w(k) \tag{2.22}$$

The resulting closed loop coefficient matrices A_{CL} and F_{CL} are defined as follows:

$$A_{CL} = A_c + B_c K_c \tag{2.21}$$

$$F_{CL} = F_c + B_c K_d \tag{2.22}$$

The analysis of the closed-loop system will be described so that the stability of the system, disturbance accommodation, and control input minimization is guaranteed.

2.3: Analysis

As previously discussed in Chapter 1, the three main objectives of the disturbance accommodation control design are to achieve closed-loop system stability, disturbance accommodation, and minimization of the control input for systems without a input term in the output equation. Each of these objectives will be considered and analyzed as a function of the pseudo-output weighting parameters (2.5) to find a specific range of acceptable values for ϕ and γ.

2.3.1: Stability

The first objective of the controller is to achieve closed-loop system stability. Since the pseudo-output introduces the two weighting parameters ϕ and γ into the system, stability can no longer be guaranteed. The stability of the system is analyzed by investigating the eigenvalues of the closed loop system. For a discrete-time system, the eigenvalues should lie within the unit circle in the complex z-plane. In order to guarantee the stability of the system, the maximum eigenvalues of the closed-loop system A_{CL} are analyzed as a function of the weighting parameters to determine the allowable range of ϕ and γ.

$$\left|\lambda(A_{CL})\right| < 1 \tag{2.23}$$

It should be noted that although achieving closed-loop stability is a primary objective in this analysis, each analysis will be performed independent of one another.

2.3.2: Disturbance Accommodation

Another objective of the controller is to accommodate the effect of the disturbance on the system. According to DAC theory, the disturbance signal must be of known waveform-type but unknown magnitude, arrival time and duration. The disturbance accommodation techniques that are used are the Disturbance Minimization techniques, as discussed in [1]. In this work, a step-type disturbance signal is applied to the system, although the results could be extended to include ramp-type disturbances or sinusoidal-type disturbances. As seen in the system model (2.1), the disturbance is applied to both the state equation and the output of the system. To quantify and analyze the disturbance accommodation, the concept of a disturbance Grammian, G_D is introduced. The disturbance Grammian is determined from the closed-loop coefficient matrix for the disturbance term, F_{CL}, as seen in the closed-loop system model (2.13), and is defined as:

$$G_D \equiv F_{CL}{}^T F_{CL} \tag{2.24}$$

Since the disturbance being used in this work is a step-type disturbance, the resulting disturbance Grammian will be a scalar value. Considering the applied step-type disturbance signal with a magnitude of 1, the disturbance Grammian G_D should be less than that magnitude of the disturbance signal to achieve disturbance attenuation:

$$|G_D| < 1 \tag{2.25}$$

By ensuring this condition will be satisfied, the controller will dampen the effect of the disturbance signal.

2.3.3: Cost Function

The last main control objective that will be discussed is the application of and analysis based on a defined cost function. After the analysis of the closed-loop stability and disturbance accommodation, an initial region of ϕ and γ parameter values from the pseudo-output equation is determined. To further determine appropriate values of these parameters, a cost function was defined. The cost function *J* is defined as the sum of the input *u(k)* squared and output *y(k)* squared over a range of the discrete time step *k*. The range of *k* is the same for each system and the disturbance signal is applied over the same range of *k*:

$$J = \sum_{k=k_{\min}}^{k_{\max}} \left(u(k)^2 + y(k)^2 \right) \tag{2.26}$$

In the analysis of this cost function, the base-10 logarithm of the cost function values was taken in order to produce values of a manageable order for illustration purposes. The analysis of the cost function is used to evaluate the minimization of the control input and output as a function of the pseudo-output weighting parameters. Evaluating the cost function in terms of ϕ and γ will allow for the selection of ϕ and γ values to assure the achievement of the control objectives with minimum values of J.

2.4: Analysis Summary

Now that the analysis of the controller for systems without a control term in the measurement equation has been presented, the stability of the closed-loop system is guaranteed along with disturbance accommodation in terms of the pseudo-output weighting parameters ϕ and γ. The union of the stability analysis results as a function of ϕ and γ and disturbance accommodation results as a function of ϕ and γ yield a range of acceptable ϕ and γ values. The values of ϕ and γ selected from within this range guarantee the stability of the closed-loop system and the disturbance accommodation of the applied disturbance signal. Evaluating the cost function as a function of ϕ and γ provides additional insight on the selection of ϕ and γ. Upon the completion of these analyses, an acceptable range of ϕ and γ values has been defined. In the following chapter and case studies, particular values of ϕ and γ are chosen to simulate the input, output, and state responses of the system.

Now that the controller design and analysis have been presented, case studies demonstrating the controller's abilities will be presented in the following chapter. Three case studies will be considered; first order systems, second order systems, and third order systems. For each case study, the development of the systems will be shown, followed by the controller development and stability analysis, disturbance accommodation analysis and cost function analysis. Following the development and analysis of each case study, the results will be discussed.

Chapter 3: Case Studies

In order to most effectively demonstrate the controller and technique proposed in the previous chapter, three different case studies will be investigated; first-order systems, second-order systems and third-order systems. For each of the first, second, and third-order system case studies, three variations will be considered to investigate the effect of system stability on the controller's performance. The first variation and most stable system in each case study considered were taken from examples in [12]. By using three systems for each case study with varying degrees of stability, trends in the performance of the controller can be investigated. The disturbance signal applied to each system is a step-type disturbance and is therefore of known waveform, but unknown magnitude, arrival time and duration. The objective of the analysis of each system is to obtain ideal pseudo-output weighting parameter values to accomplish the control objectives of closed-loop stability, disturbance attenuation and control input minimization.

3.1: First-Order Case Study

The first-order case study will consist of the investigation of three discrete-time example systems with varying degrees of stability, System 1-I, System 1-II, and System 1-III. As previously mentioned, stability is determined by the location of the eigenvalues or poles of the system's transfer function within the unit circle.

3.1.1: System Development

For the first-order example systems, the system model is described by the following state, measurement and disturbance model equations.

$$x(k+1) = \alpha x(k) + bu(k) + fw(k) \tag{3.1.1}$$

$$y(k) = cx(k) + gw(k) \tag{3.1.2}$$

$$w(k+1) = ew(k) + \sigma(k) \tag{3.1.3}$$

Since first-order examples are being considered, the weighting coefficients for the system model in (3.1.1) will be scalar values and will therefore be in canonical form, as is required by the proposed control technique. The scalar coefficients for the first-order example systems can be seen in Table 1.

Coefficients	System 1-I	System 1-II	System 1-III
α	0.5	1.1	1.3
b	1	1	1
f	1	1	1
c	1	1	1
g	0.064	0.064	0.064
e	1	1	1

Table 1: System Parameters

The first example system to be considered is System 1-I and has α=0.5, representing a stable eigenvalue. Following, System 1-II is slightly unstable with its eigenvalue located just outside of the unit circle. The last first-order system to be considered is the most unstable system, System 1-III, with an eigenvalue of 1.3. The location of the eigenvalues with respect to the unit circle for each system can be seen in Figure 3.1.1, where red represents System 1-I, blue represents System 1-II, and green represents System 1-III, respectively. As the eigenvalues move farther outside the unit circle, the open loop system is less stable.

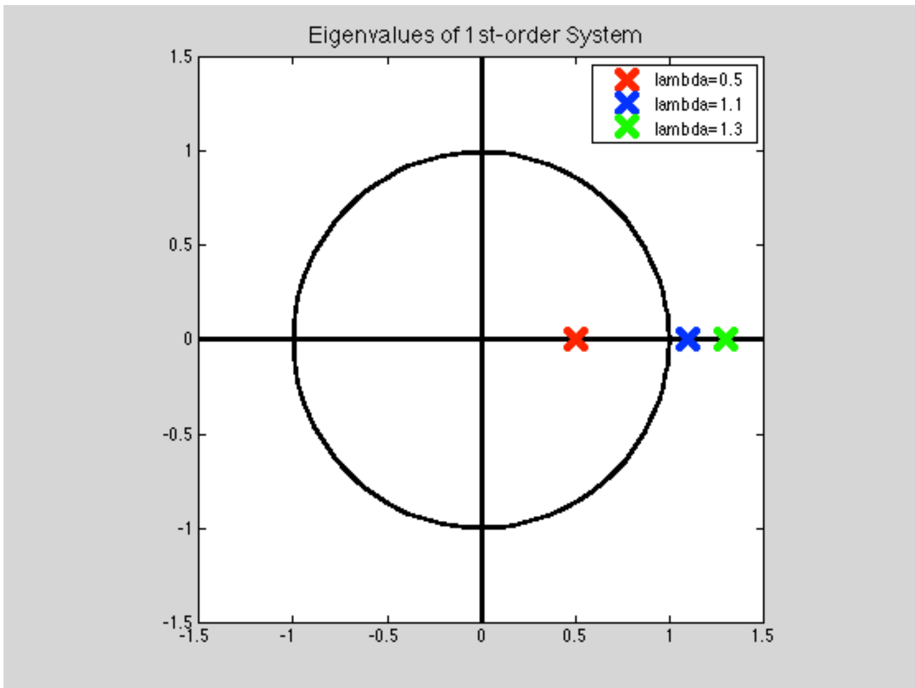

Figure 3.1. 1: Eigenvalues of 1st order open loop systems

Even though the open-loop system might not be stable, the objectives of this work are such that upon the application of the controller, a stable closed-loop system will result.

For each System 1-I, System 1-II, and System 1-III, the pseudo output z(k) is introduced to provide a control term in the measurement equation as mentioned in (2.4). Following the procedure outlined in Chapter 2, the composite system is composed of the state update equation and the pseudo-output update equation.

$$\begin{bmatrix} x(k+1) \\ z(k+1) \end{bmatrix} = A_p \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} + B_p u(k) + F_p w(k) \tag{3.1.4}$$

Since the original first order system has now been augmented with the pseudo-output equation, the resulting weighting matrices of the composite system are of second-order.

$$A_p = \begin{bmatrix} \alpha & 0 \\ \phi c \alpha & 0 \end{bmatrix}, \quad B_p = \begin{bmatrix} b \\ \phi cb + \gamma \end{bmatrix}, \quad F_p = \begin{bmatrix} f \\ \phi(cb + ge) \end{bmatrix} \tag{3.1.5}$$

It is important to note that the weighting coefficients A_p, B_p, and F_p are now in terms of the pseudo-output weighting coefficients ϕ and γ. Now that each of the systems have been developed, the controller design will be discussed. The objective of the controller design and analysis is to obtain ideal values of ϕ and γ that will result in closed-loop system stability, disturbance attenuation, and control input minimization.

3.1.2: Controller Design

The controller is designed according to the process outlined in Chapter 2, and is composed of a component to control the state as well as a component to control the disturbance (2.8). The resulting controller gain K_c and the disturbance gain K_d are determined for System 1-I, System 1-II, and System 1-III. Following the controller development in Chapter 2, the gains for the first order systems are shown below.

$$K_c = (B_p{}^T B_p)^{-1} B_p{}^T A_p \tag{3.1.6}$$

$$K_d = (B_p{}^T B_p)^{-1} B_p{}^T F_p \tag{3.1.7}$$

The resulting controller gains for each first order system are shown in Table 2.

	System 1-I	System 1-II	System 1-III
K_c	$\begin{bmatrix} \frac{-0.5(1+\phi(\phi+\gamma))}{1+(\phi+\gamma)^2} & 0 \end{bmatrix}$	$\begin{bmatrix} \frac{-1.1(1+\phi(\phi+\gamma))}{1+(\phi+\gamma)^2} & 0 \end{bmatrix}$	$\begin{bmatrix} \frac{-01.3(1+\phi(\phi+\gamma))}{1+(\phi+\gamma)^2} & 0 \end{bmatrix}$
K_d	$\frac{1+1.064\phi(\phi+\gamma)}{1+(\phi+\gamma)}$	$\frac{1+1.064\phi(\phi+\gamma)}{1+(\phi+\gamma)}$	$\frac{1+1.064\phi(\phi+\gamma)}{1+(\phi+\gamma)}$

Table 2: First-order Control and Disturbance Gains

Once the controller and disturbance gains have been determined, the controller can be applied to the system, resulting in the closed-loop system. The closed-loop system can be seen in (2.13) of Chapter 2. The closed-loop system matrix coefficients A_{CL} and F_{CL} are going to be used to evaluate closed-loop stability and disturbance accommodation. The eigenvalues of A_{CL} will be used to analyze the stability of the systems, while F_{CL} will be used to calculate the disturbance Grammian and quantifying the disturbance attenuation.

3.1.3: Analysis

In this section, the three main controller objectives will be analyzed for each of the first-order systems. To investigate the stability of the systems, the closed-loop eigenvalues will be analyzed as a function of ϕ and γ, the pseudo-output weighting parameters. In order to guarantee closed-loop stability, the closed loop eigenvalues must remain within the unit circle. Since the closed-loop system coefficient matrices are a function of ϕ and γ, only those parameter values resulting in closed-loop eigenvalues with a magnitude of less than one are suitable to achieve stability. To best determine the ideal range for ϕ and γ, each system is analyzed for values from 0 to 20. Only positive values of ϕ and γ were used because of conditions that will be discussed in Chapter 4: Considerations for Implementation. In short, the pseudo-output weighting parameters are taken to be greater than zero to ensure that the conditions for stability are satisfied. To ensure disturbance attenuation, the disturbance Grammian is calculated from the closed-loop disturbance matrix coefficient, F_{CL} and analyzed to make sure its magnitude will be less than one for the specified range of ϕ and γ values. Once a region of acceptable values to achieve stability and disturbance accommodation has been found, the cost function in (2.20) will be analyzed. Upon performing the cost function analysis, the state trajectories for each system are investigated for particular values of ϕ and γ.

System 1-I Evaluation

The first system to be considered for analysis is System 1-I, the stable first-order system. The closed-loop eigenvalues are investigated as a function of ϕ and γ and the resulting figure is shown below.

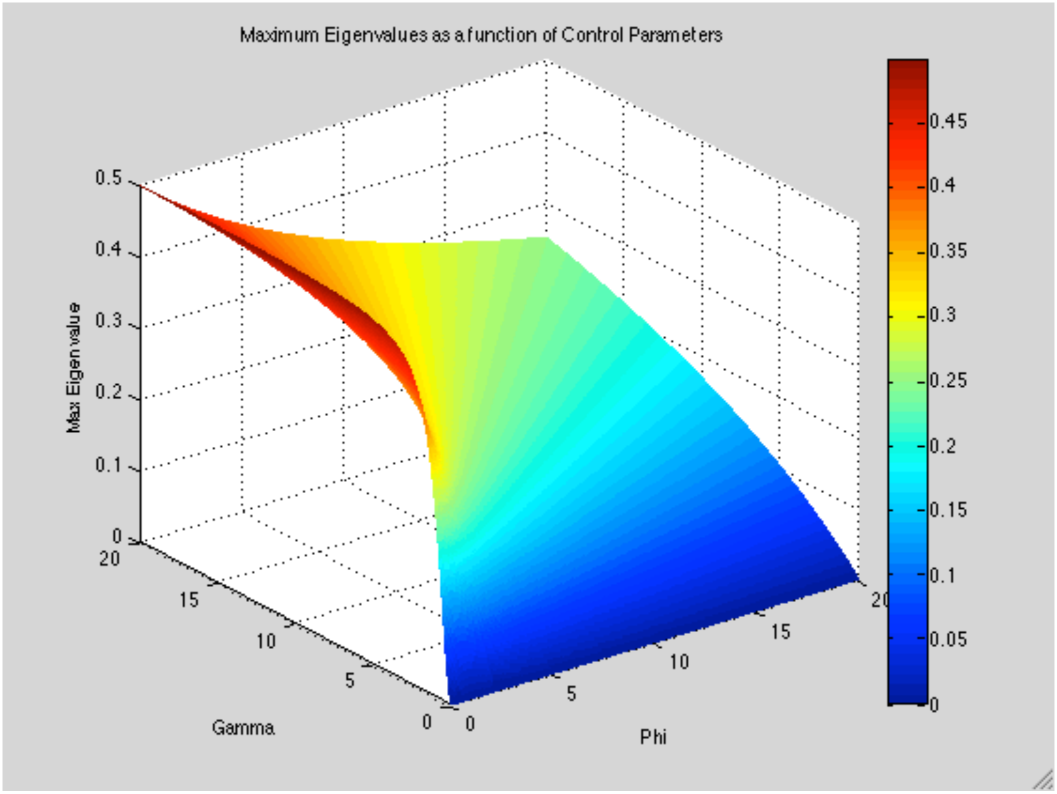

Figure 3.1. 2 Maximum Eigenvalues as a Function of Control Parameters for System 1-I

It is noticeable in this figure that for any value of ϕ and γ from 0 to 20, the system will remain stable. This guaranteed stability is due to the fact that $|\lambda_{max}| < 1$ is true over the whole region. This result is to be expected, considering System 1-I was stable in the open-loop case.

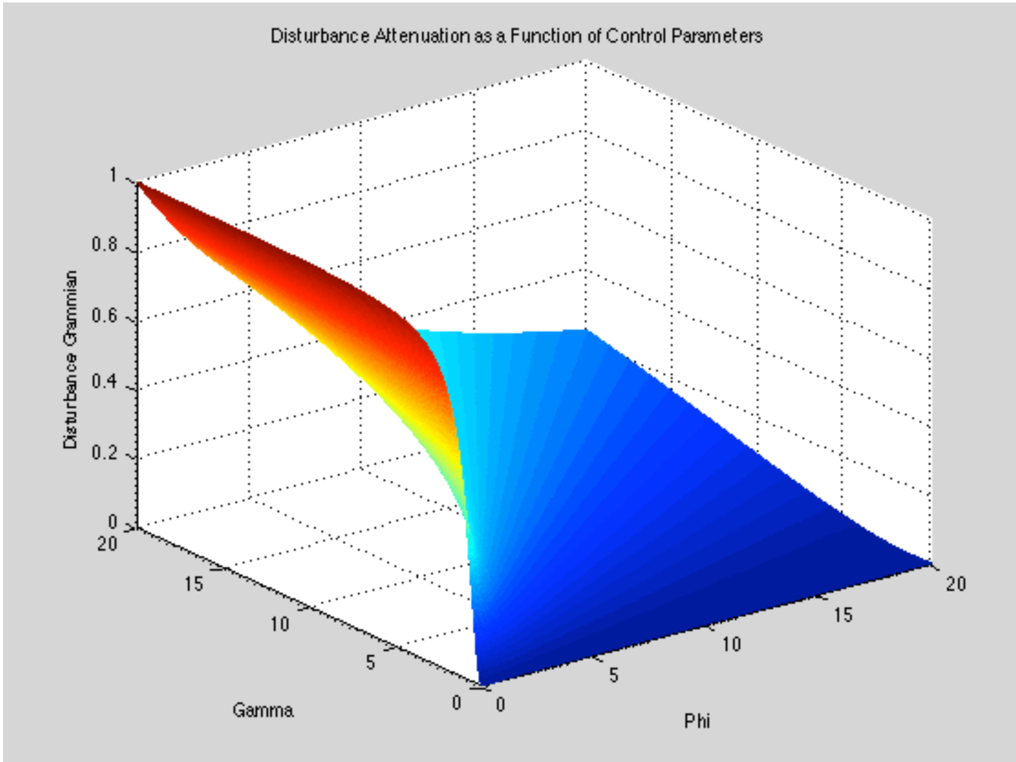

Figure 3.1. 3 Disturbance grammian versus ϕ and γ for System 1-I

To evaluate the disturbance attenuation, Figure 3.1.3 shows the magnitude of the disturbance Grammian G_D versus the pseudo-output weighting parameters, ϕ and γ. As seen in the figure, the disturbance Grammian is less than one, although at the limit of γ, the disturbance Grammian is approaching the value of one. This result indicates that for any ϕ and γ from 0 to 20, $|G_D| < 1$ therefore, the disturbance will be attenuated.

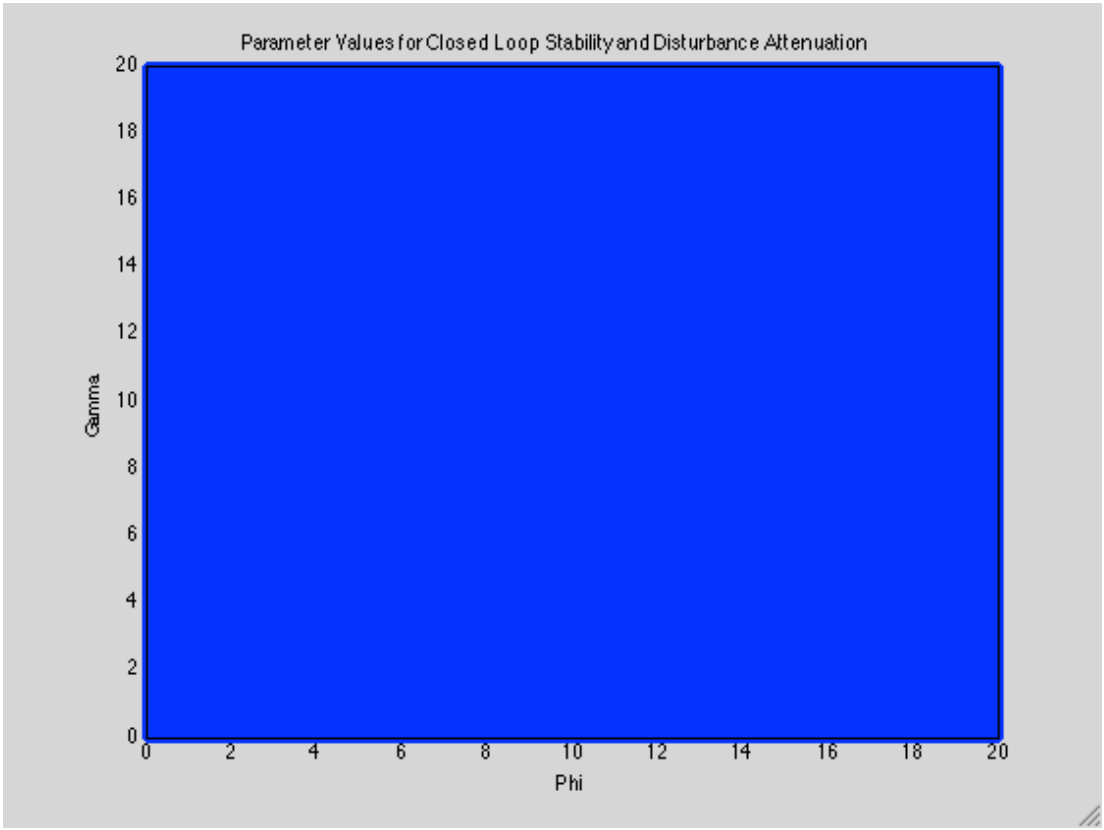

Figure 3.1. 4 Closed Loop stability and disturbance accommodation range of ϕ and γ for System 1-I

Now that the stability and disturbance attenuation have been investigated, Figure 3.1.4 displays the range of ϕ and γ values where both the closed-loop system is stable and the disturbance is accommodated. As seen in this figure, any particular value of ϕ and γ from 0 to 20 will result in a stable closed-loop system with disturbance accommodation. This result is to be expected considering the results of the stability and disturbance accommodation analyses for System 1-I.

To further the investigation of acceptable ϕ and γ parameter values, the performance cost function is analyzed as a function of ϕ and γ. The cost associated with particular parameter values is represented by the various colors, where warmer colors represent a higher cost associated with using that particular pair of values, and cooler colors represent a lower cost.

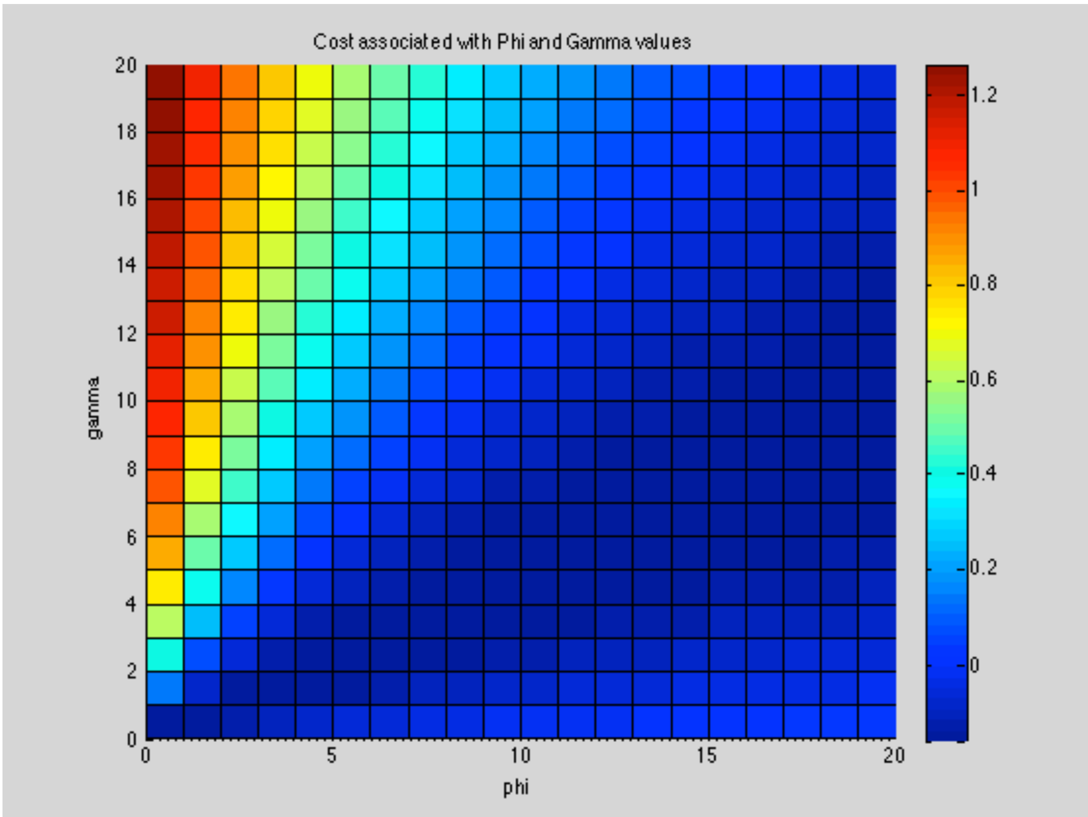

Figure 3.1. 5 Cost Function Analysis System 1-I

This figure shows the cost associated with each value of ϕ and γ, where the cost once again is defined as $J = \sum_{k=k_{\min}}^{k_{\max}} \left(u(k)^2 + y(k)^2 \right)$. It can be seen that a higher cost results for higher values of γ and lower values of ϕ, as seen by the shades of red in Figure 3.1.5. This region also corresponds to the region in Figure 3.1.3, where the disturbance Grammian was approaching a magnitude of one.

Now than acceptable region of the pseudo-output weighting parameters has been defined, the state trajectories can be investigated. Three sets of ϕ and γ values were chosen to simulate the trajectories of System 1-I. This was done to see the differences in responses as ϕ is increased and γ remains constant, and when γ is increased and ϕ remains constant.

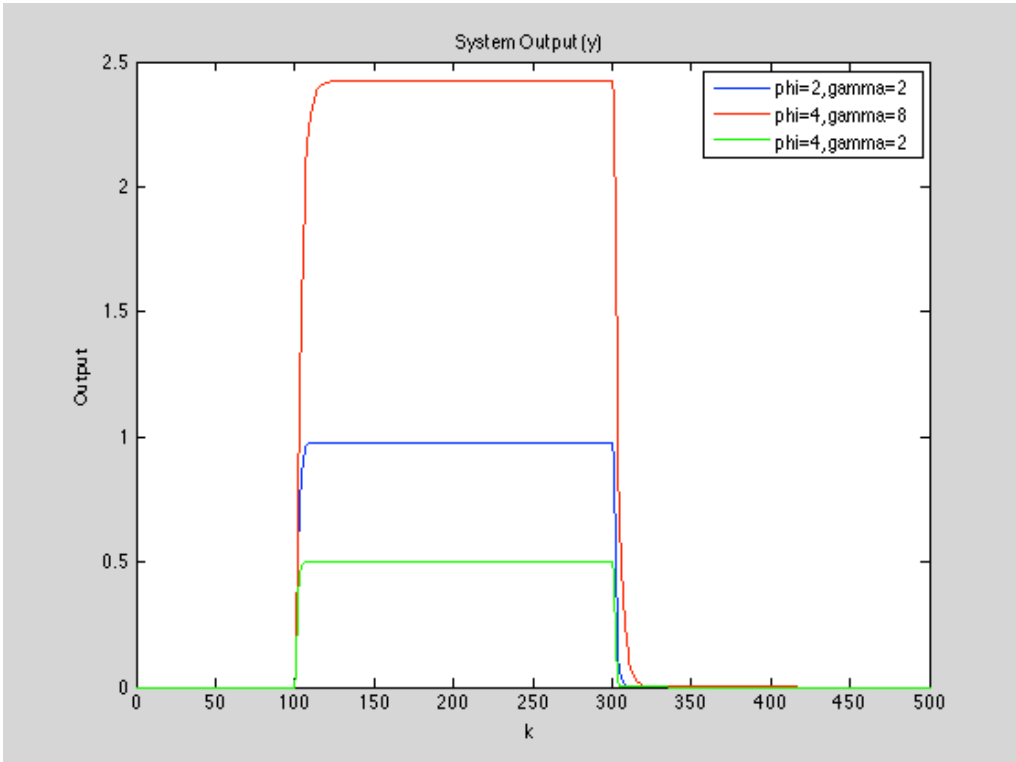

Figure 3.1. 6 Output for System 1-I

Figure 3.1.6 shows the system output versus the sampling instant k for particular values of ϕ and γ chosen with guidance from the previous cost function analysis, Figure 3.1.5. It is noticeable in this figure that as ϕ increases while γ remains constant, the magnitude of the output response decreases. On the other hand, while γ is increased and ϕ kept constant, the magnitude of the output response increases. The next figure shown is the control input of the system versus the sampling instant k.

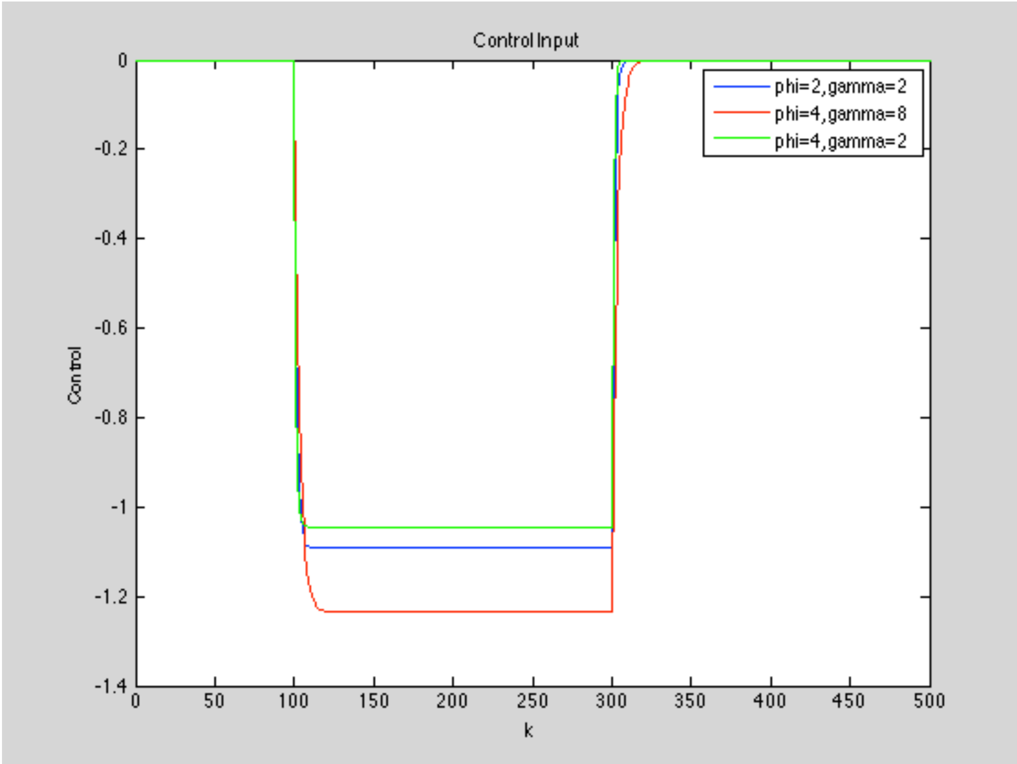

Figure 3.1. 7: Input for System 1-I

It can be seen in this figure that as ϕ is increased and γ remains constant, the magnitude of the input decreases, and while γ is increased and ϕ remains constant, the magnitude of the input increases.

Following the control input response, the state variables of System 1-I are shown, where the second state variable, x_2, represents the pseudo-output z(k) from the composite system.

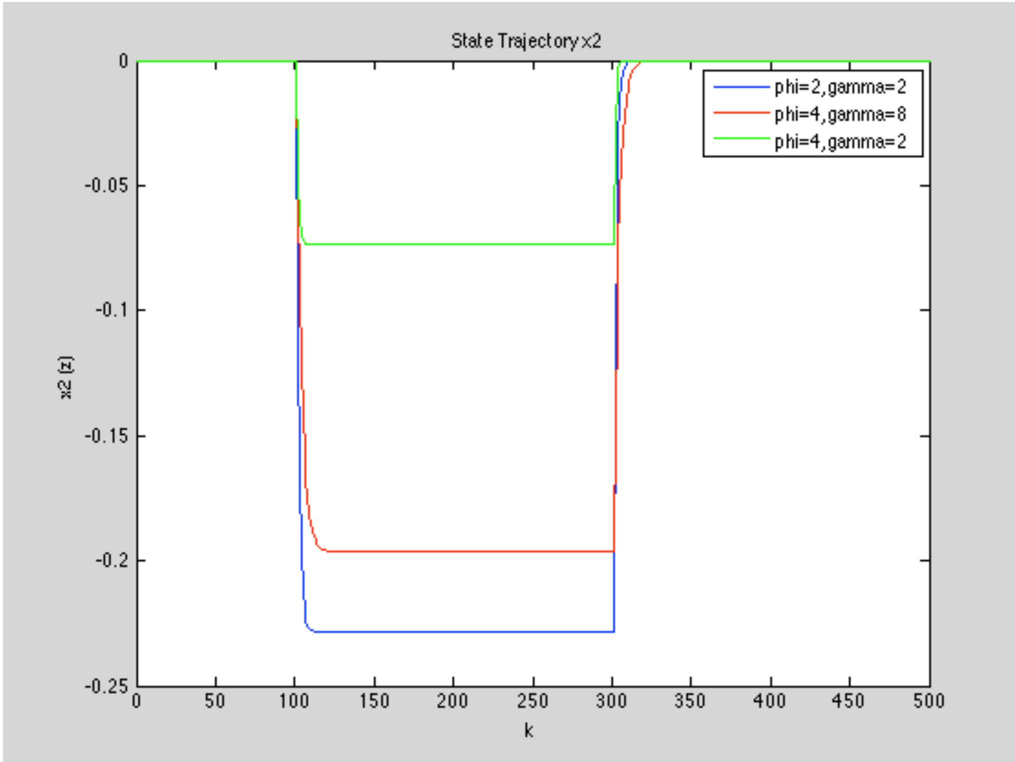

Figure 3.1. 8: State trajectory x_2 for System 1-I

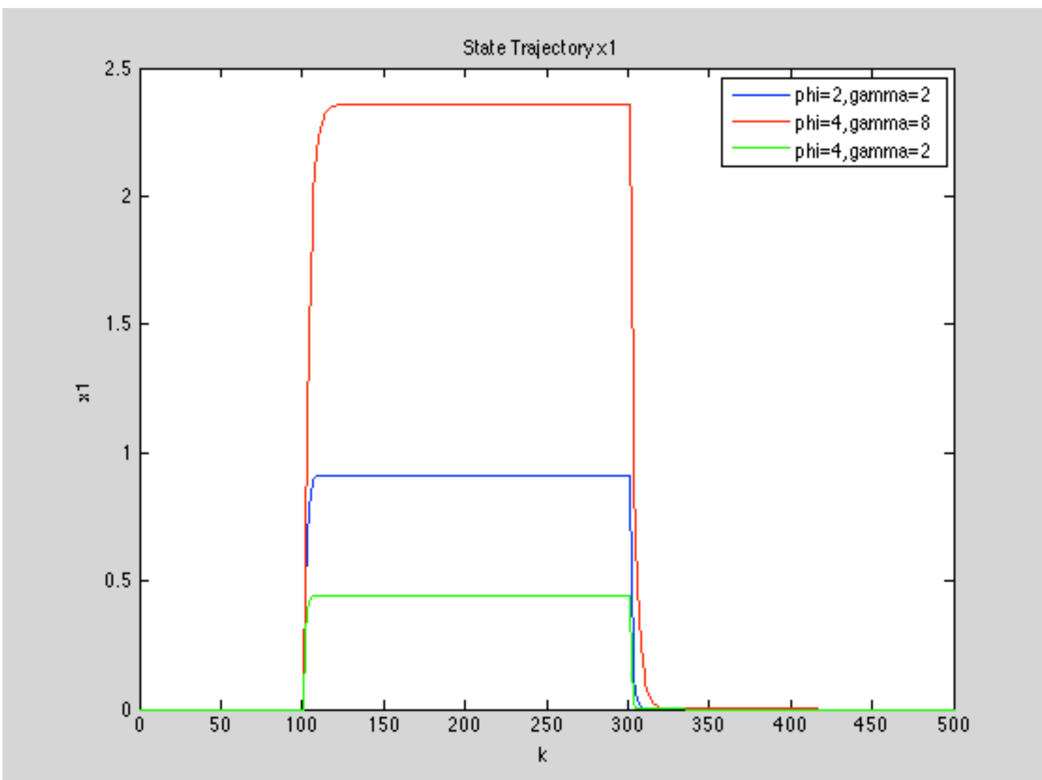

Figure 3.1. 9 State trajectory x_1 for System 1-I

Analyzing these state trajectory responses, it is seen that as ϕ increases and γ is held constant, the magnitude of the state variables decreases. On the other hand, as ϕ is held constant and γ is increased, the magnitude of the state variables increases.

System 1-II Evaluation

The second system in the first-order systems case study, System 1-II has its eigenvalue just outside of the unit circle. Proceeding through the same analysis as for System 1-I, the eigenvalues of the closed-loop system and disturbance Grammian versus ϕ and γ are shown in the following two figures, followed by the allowable region of ϕ and γ to achieve those objectives.

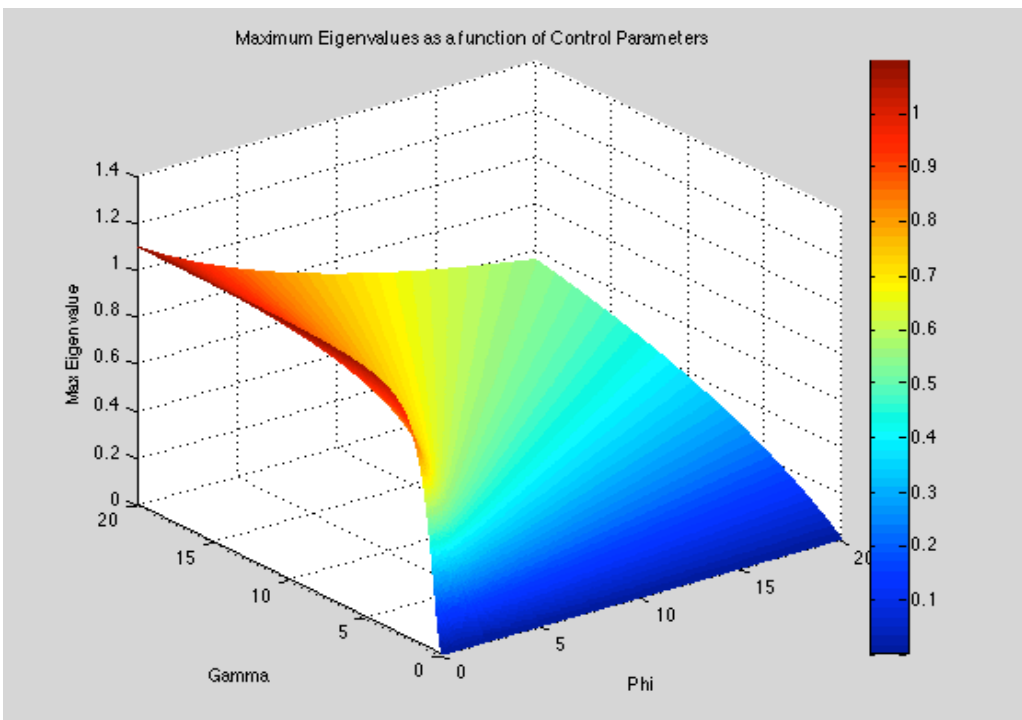

Figure 3.1. 10 System 1-II Max closed-loop Eigenvalue vs ϕ and γ

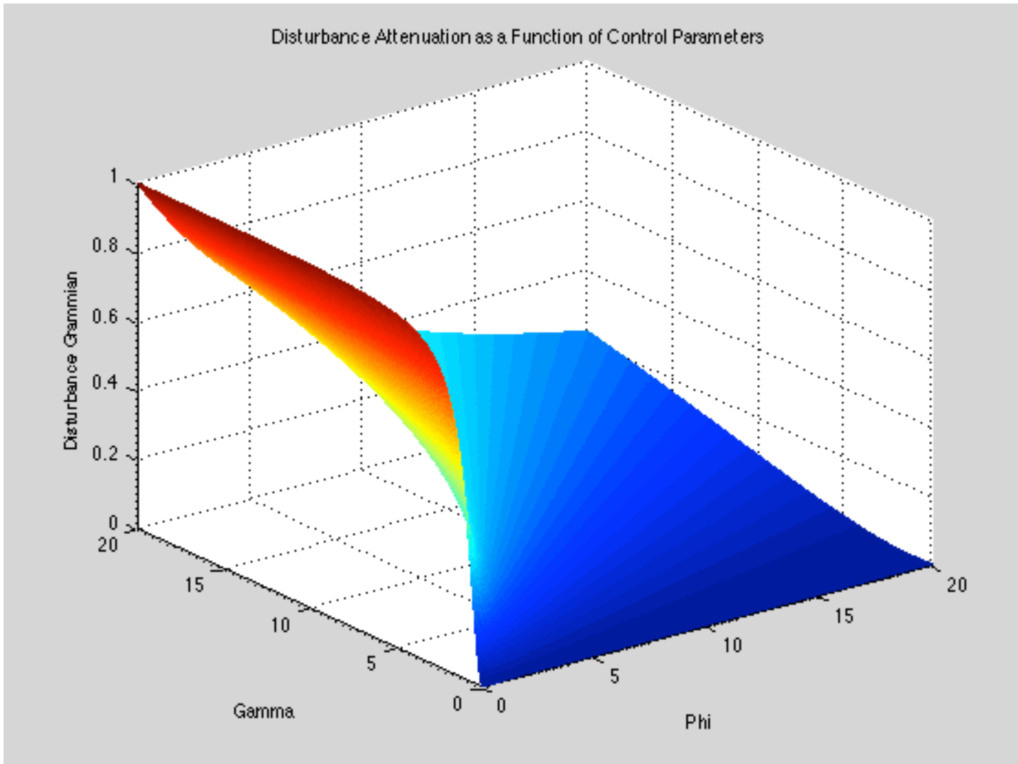

Figure 3.1. 11 System 1-II Disturbance Grammian versus ϕ and γ

It can be seen in Figure 3.1.10 that the maximum eigenvalues reach a value greater than one for values of γ reaching its upper limit, nearing 20. Despite the disturbance being accommodated for every value of ϕ and γ from 0 to 20 as seen in Figure 3.1.11, the closed-loop system will not be stable for values of γ greater than 3.16 when ϕ is zero according to Figure 3.1.10. Thus for System 1-II, the acceptable range of ϕ and γ values to achieve closed-loop stability and disturbance accommodation will decrease in size compared to the previous System 1-I.

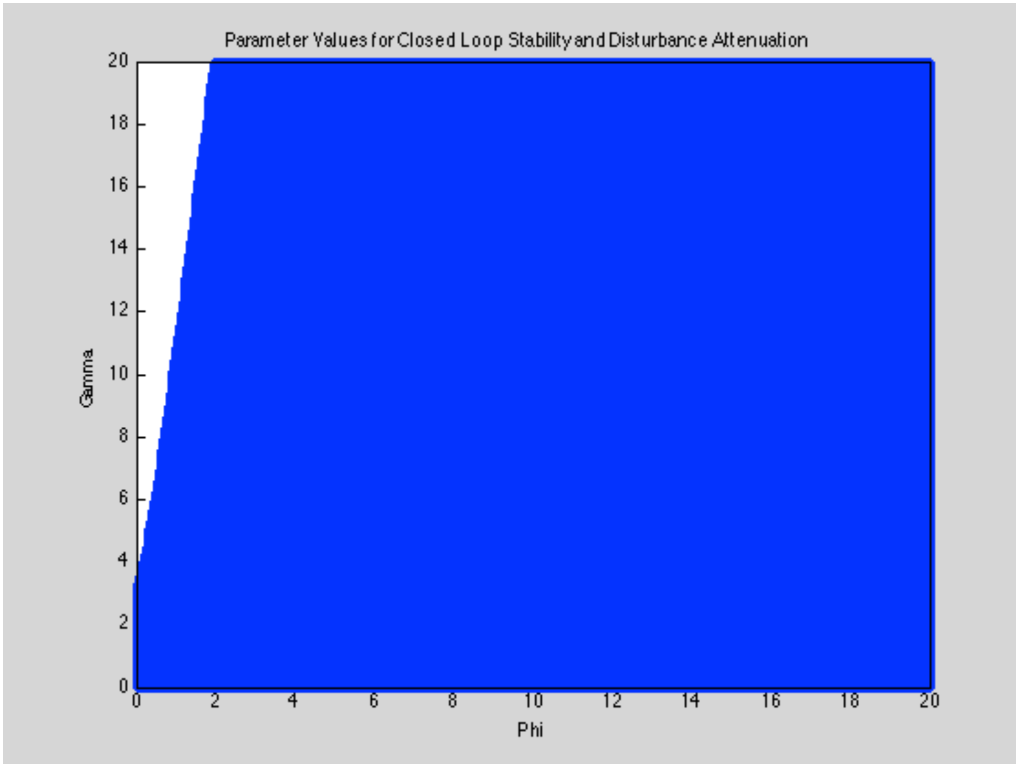

Figure 3.1. 12: Closed Loop stability and disturbance accommodation range of ϕ and γ for System 1-II

As expected, the acceptable range of ϕ and γ values in Figure 3.1.12 which shows the range of ϕ and γ for which both the eigenvalues of the closed loop system and the disturbance grammian is less than one, is indeed smaller than the previously investigated System 1-I. It can be noted that the limit of the acceptable range of values for phi and gamma does indeed start when γ is greater than three, and ϕ is equal to zero as seen in the stability analysis (Figure 3.1.10).

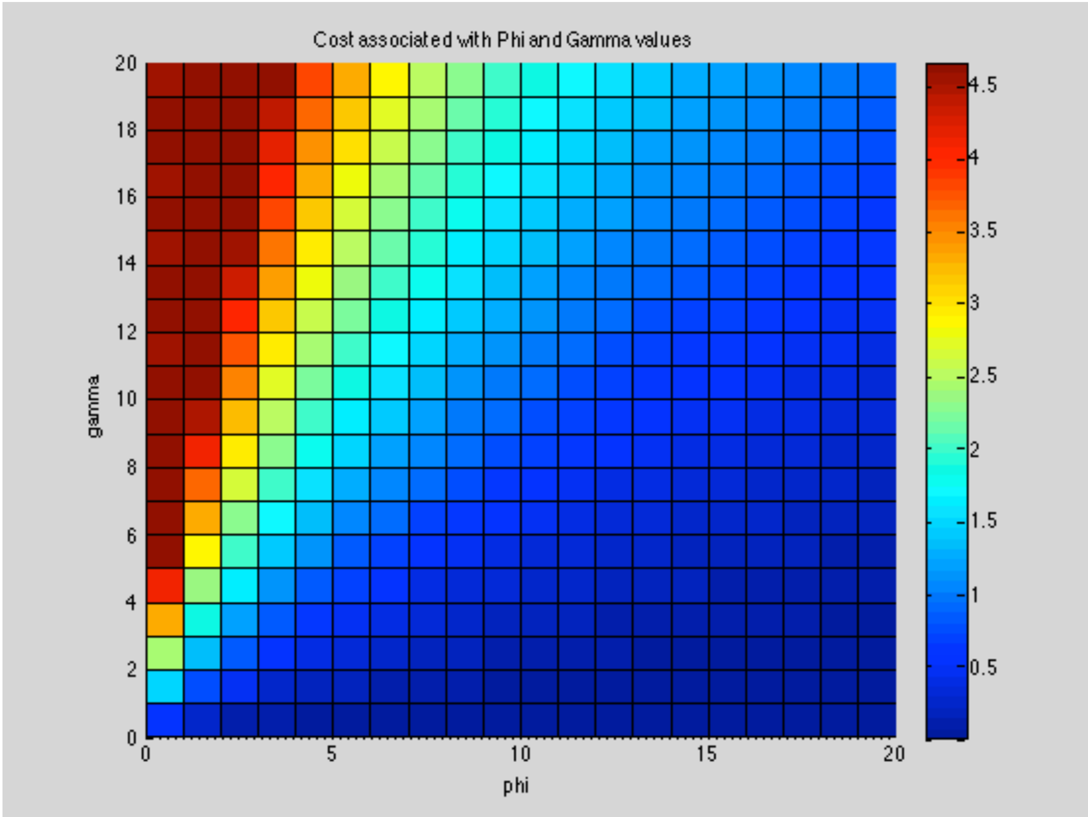

Figure 3.1. 13: Cost Analysis System 1-II

Figure 3.1.13 shows the cost analysis results for System 1-II. Considering the acceptable region of ϕ and γ values for System 1-II in Figure 3.1.12, the region of values resulting in a stable closed-loop system and disturbance accommodation is getting smaller. Considering that System 1-II is slightly unstable, it requires a higher control input to achieve stability and disturbance accommodation.

The output y(k), control input u(k), and state variables are shown below for System 1-II. As before, three particular ϕ and γ values were taken to simulate the trajectories and similar trends as mentioned in System 1-I can be observed.

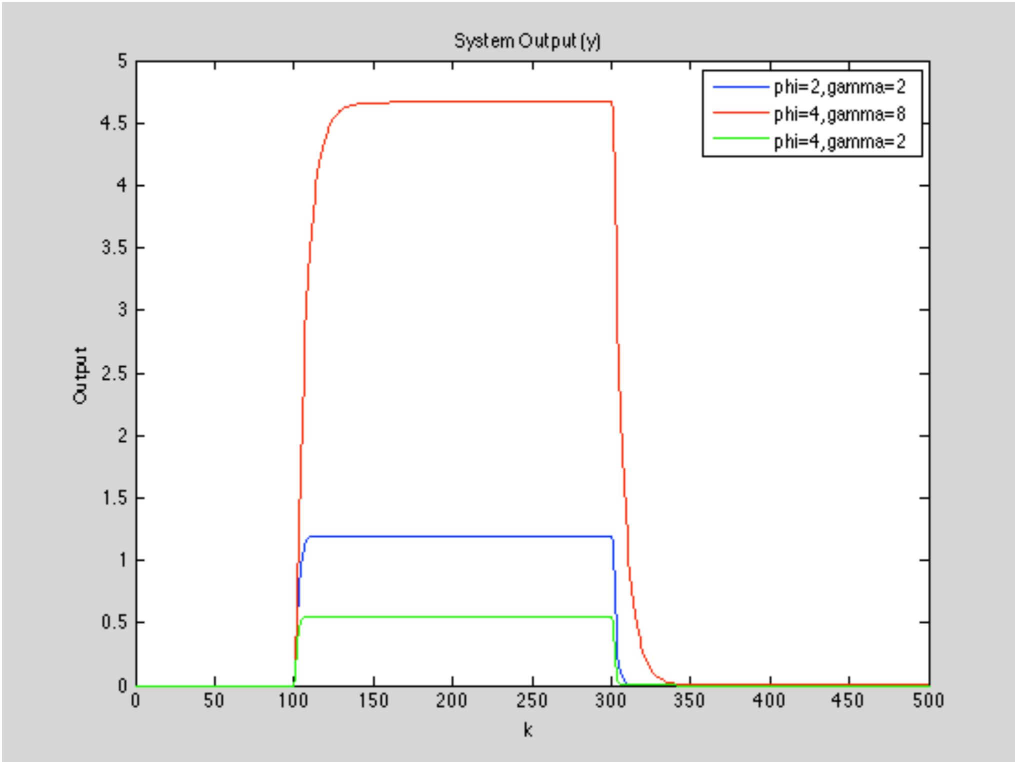

Figure 3.1. 14- System 1-II Output

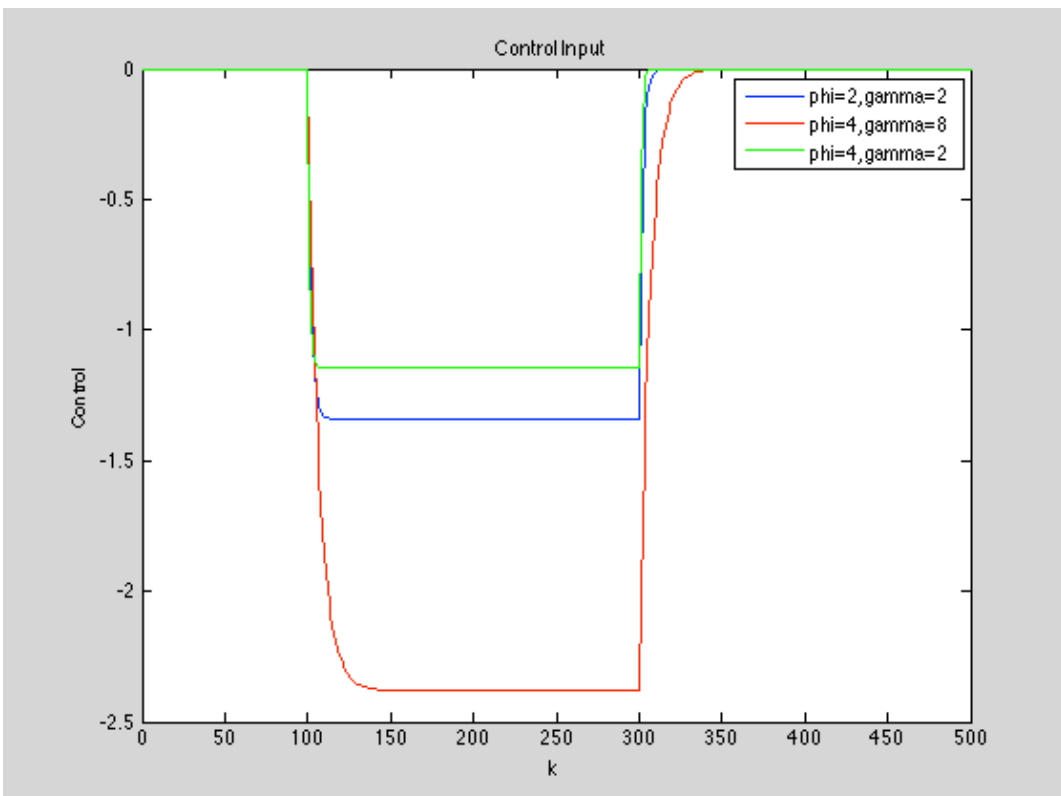

Figure 3.1. 15: System 1-II Input

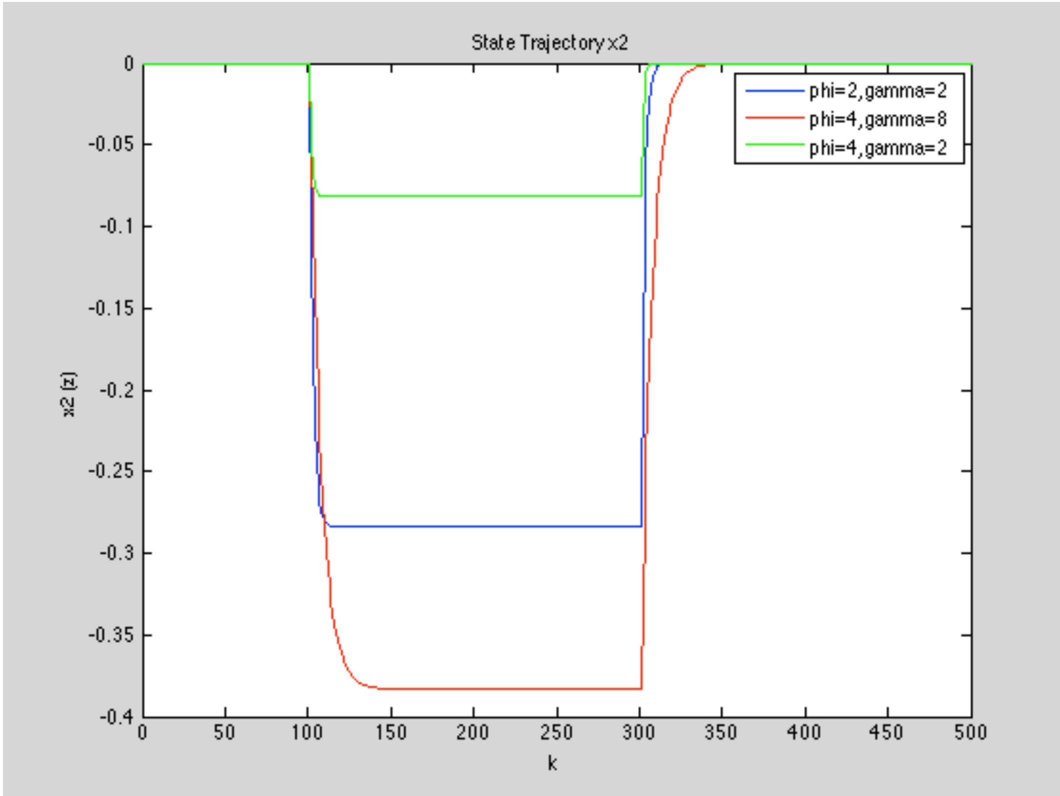

Figure 3.1. 16- System 1-II State Trajectories

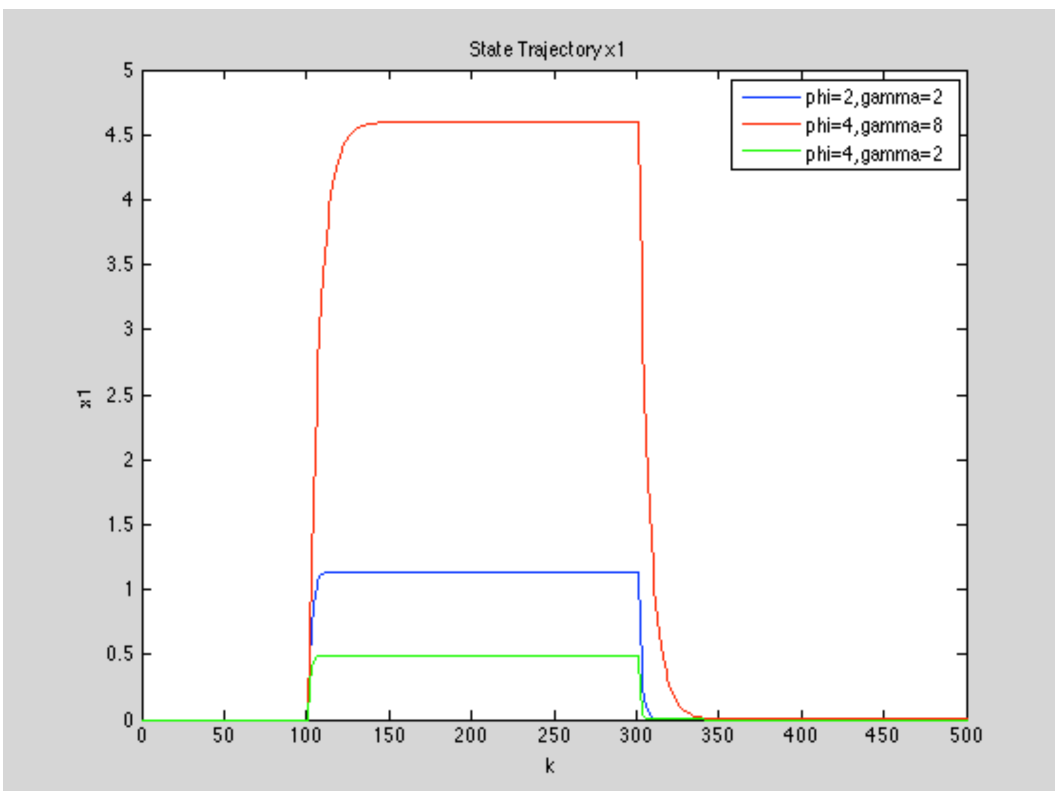

Figure 3.1. 17: System 1-II State Trajectories

The responses in these figures show that as ϕ increases and for constant γ, the magnitude of the output decreases while the magnitude of the input increases. Inversely, as γ increases and ϕ remains constant, it is observed that the magnitude of the output increases while the magnitude of the input decreases. The magnitudes of the input and output responses can be confirmed by considering the state equations of the system. For example, the magnitude of the output $y(k)$ is composed of the state variable $x_1(k)$ and the weighted disturbance term $Gw(k)$, as seen in (3.1.2).

System 1-III Evaluation

The third and final system being considered for the first-order case studies is System 1-III, with its eigenvalue farthest outside of the unit circle. The following plots show the results of the stability analysis, disturbance accommodation analysis, and the resulting acceptable ϕ and γ region. The cost analysis and System 1-III state trajectories will be observed for System 1-III. The subsequent two figures show the values of the maximum eigenvalues of the closed-loop system and the magnitude of the disturbance Grammian over the range of ϕ and γ from 0 to 20.

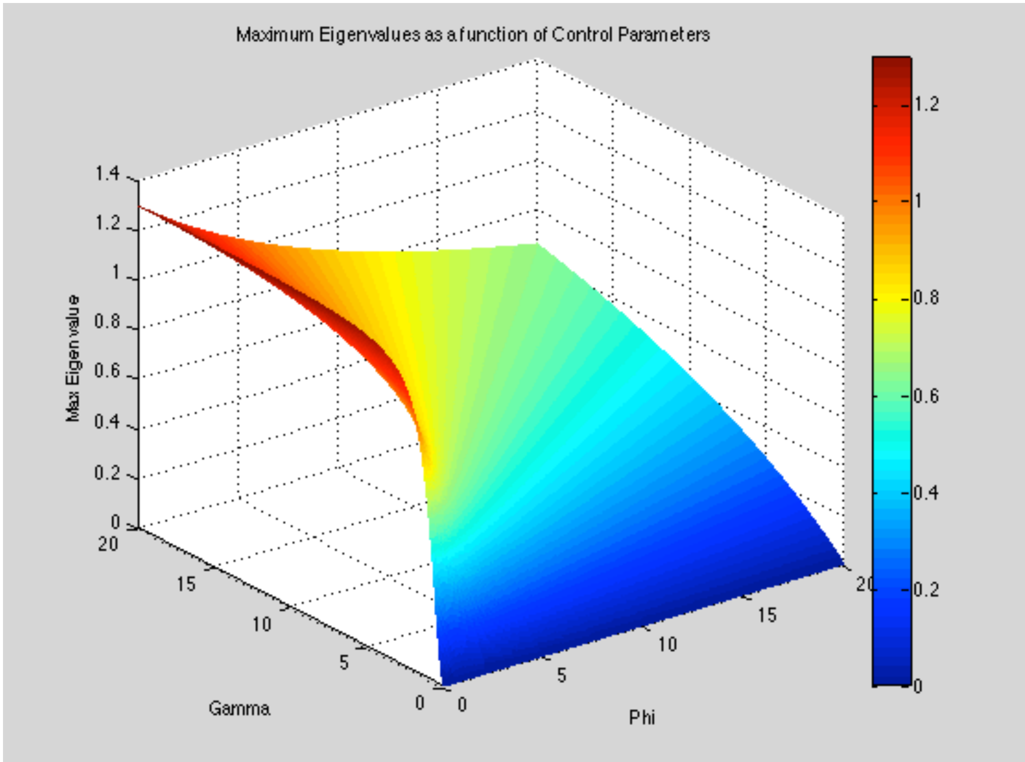

Figure 3.1. 18: System 1-III Maximum Eigenvalues versus ϕ and γ

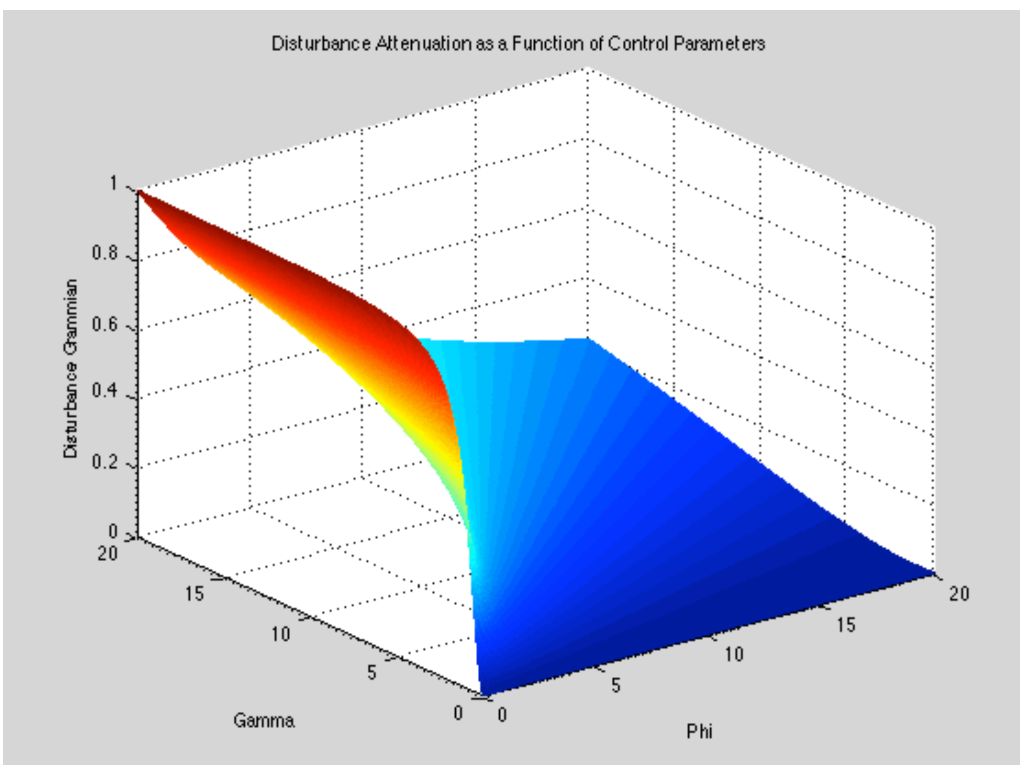

Figure 3.1. 19: System 1-III Disturbance Grammian versus ϕ and γ

It can be seen in Figure 3.1.18 that the maximum closed-loop eigenvalue of System 1-III is approximately 1.3, greater than the maximum eigenvalue for System 1-II. In other words, there is a greater range of parameter values that will not achieve closed-loop stability. According to simulations, a value of γ= 1.8 and ϕ=0 will result in an eigenvalue greater than one. Although there are less values of ϕ and γ that will yield a stable closed-loop system, all of the simulated values will achieve disturbance accommodation in Figure 3.1.19, although as γ approaches 20, the disturbance Grammian approaches one, indicating less disturbance attenuation.

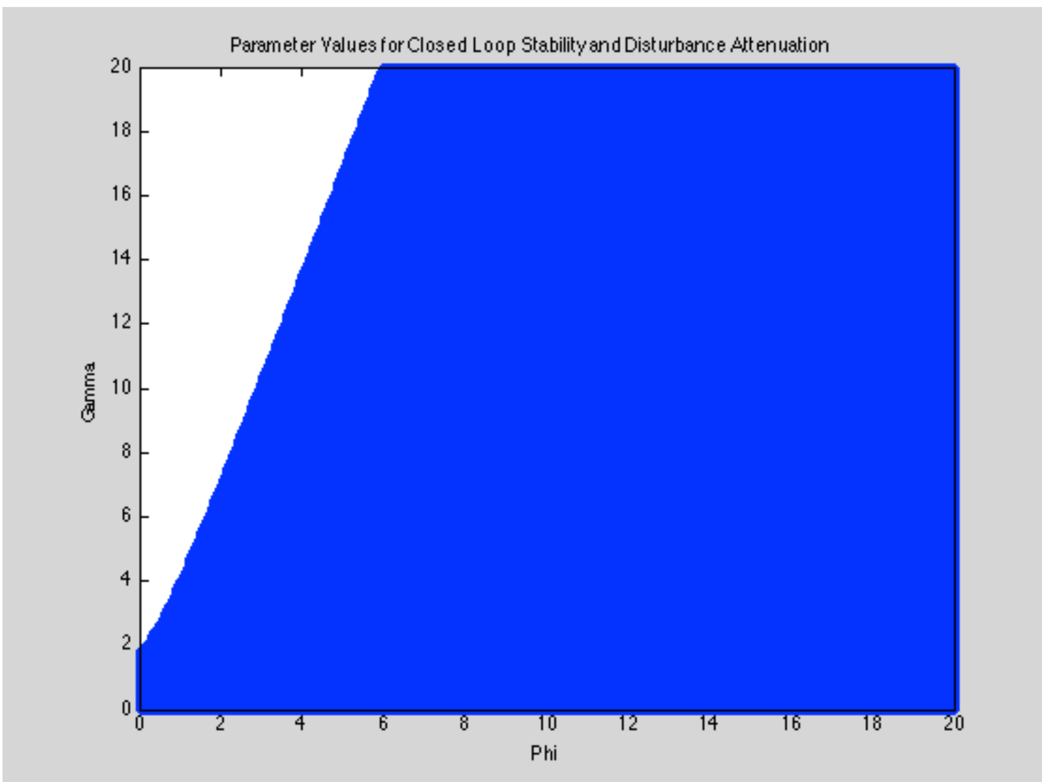

Figure 3.1. 20 :Closed Loop stability and disturbance accommodation range of ϕ and γ for System 1-III

Figure 3.1.20 shows the acceptable region of ϕ and γ values to achieve both closed-loop stability and disturbance accommodation. Following the patterns observed with System 1-I and System 1-II, there is a smaller region of acceptable parameter values along with higher costs associated with the simulated values of ϕ and γ, since the eigenvalue of System 1-III is the farthest outside of the unit circle.

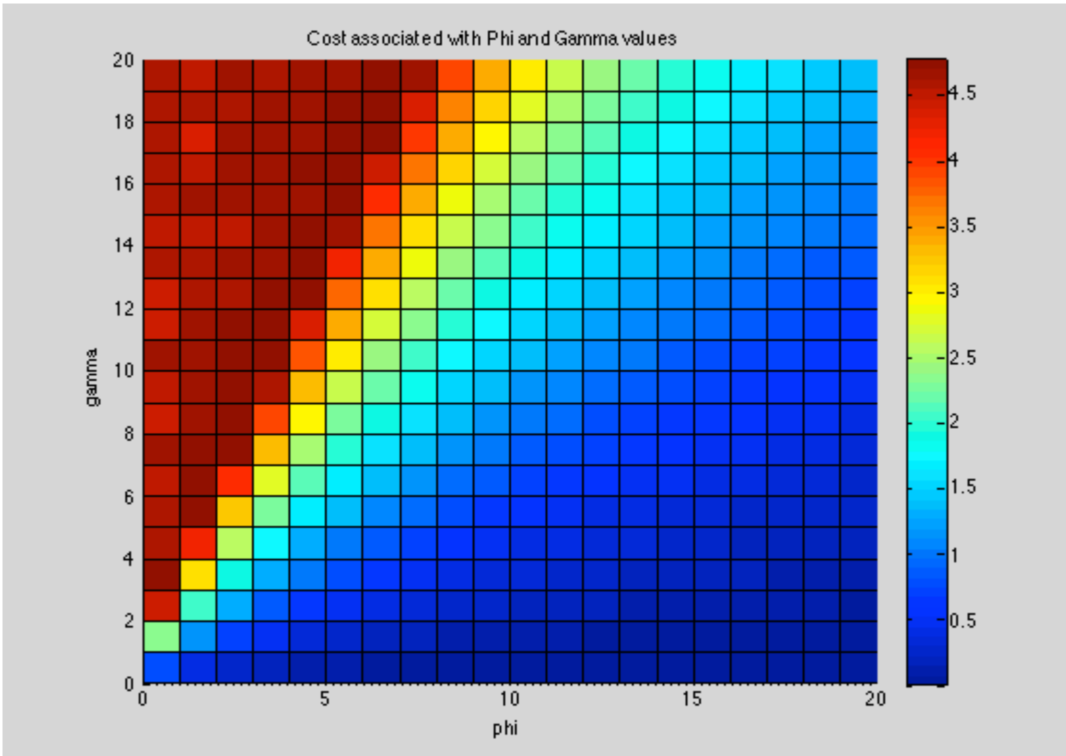

Figure 3.1. 21 Cost Analysis for System 1-III

There is a noticeable difference in Figure 3.1.21 when analyzing the cost function versus ϕ and γ comparing System 1-III to Systems 1-I and 1-II. It can be seen that the region of highest cost has grown. This is to be expected, as System 1-III represents the most unstable system. Following the cost analysis for System 1-III, the state trajectories for the system will are shown in the figures below.

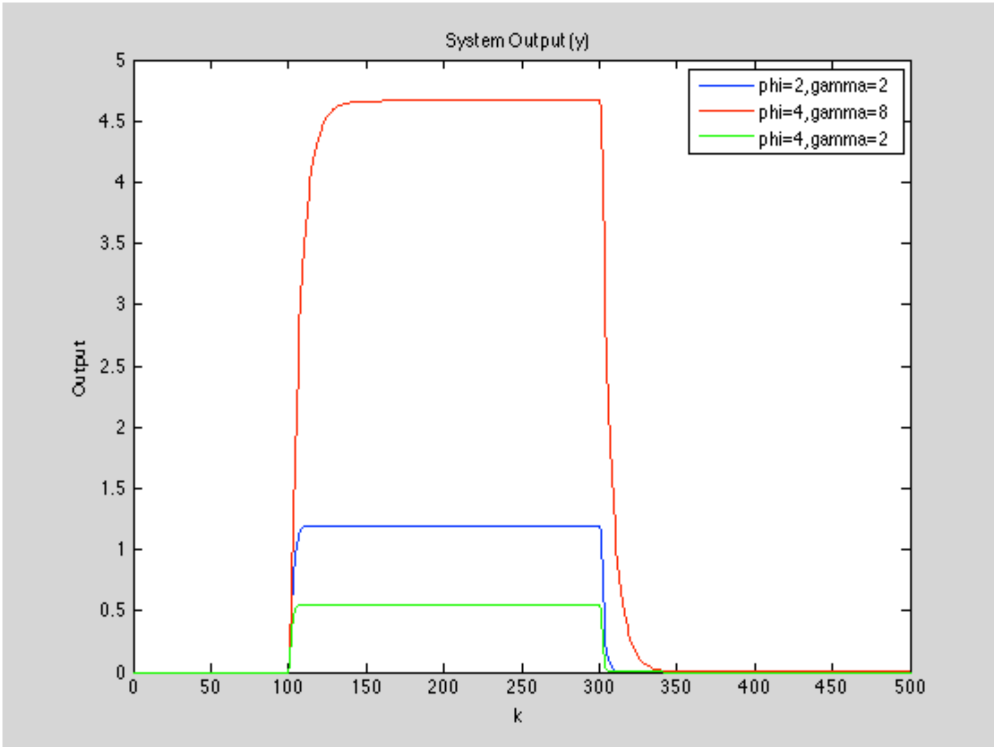

Figure 3.1. 22 Output for System 1-III

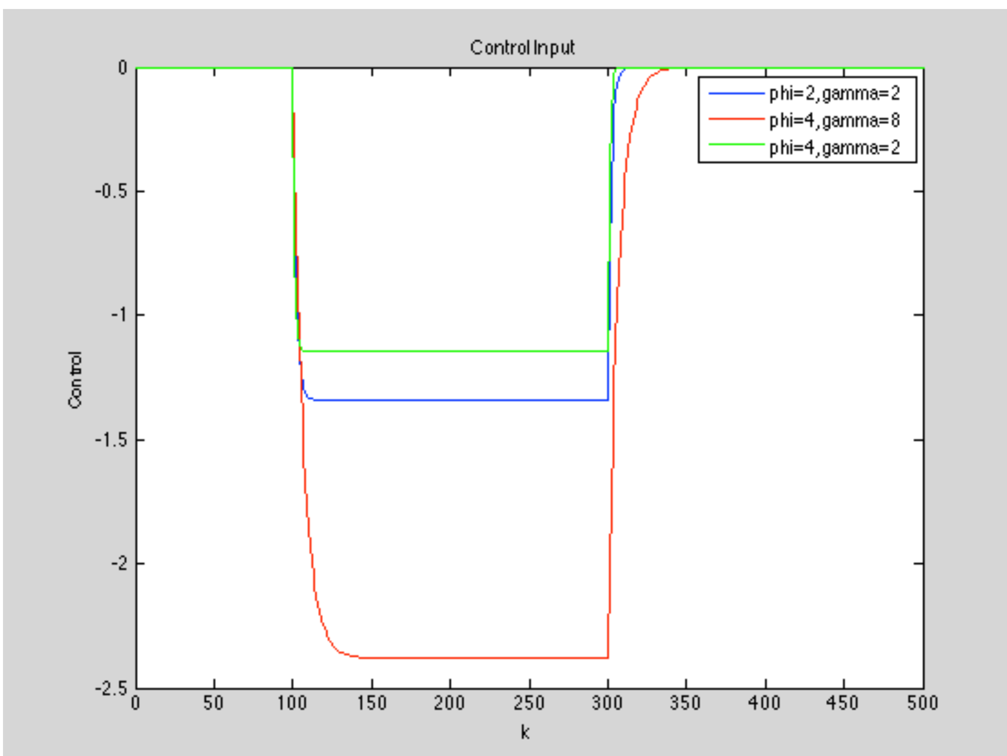

Figure 3.1. 23 Control Input for System 1-III

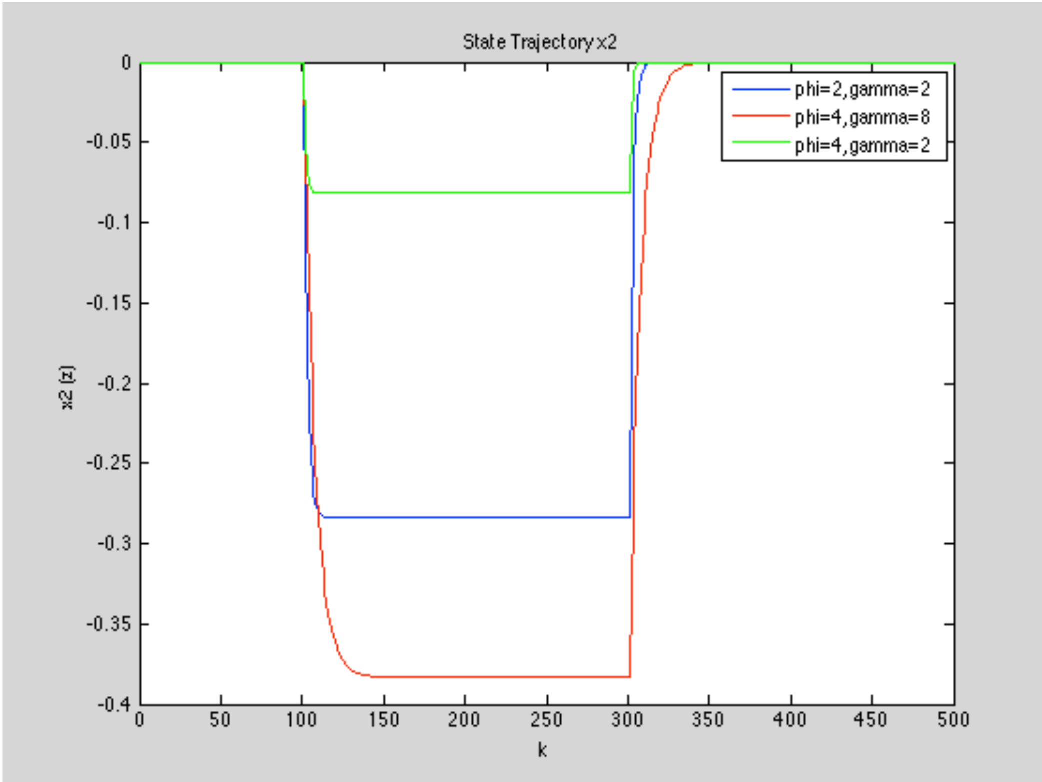

Figure 3.1. 24 System 1-III State Trajectories

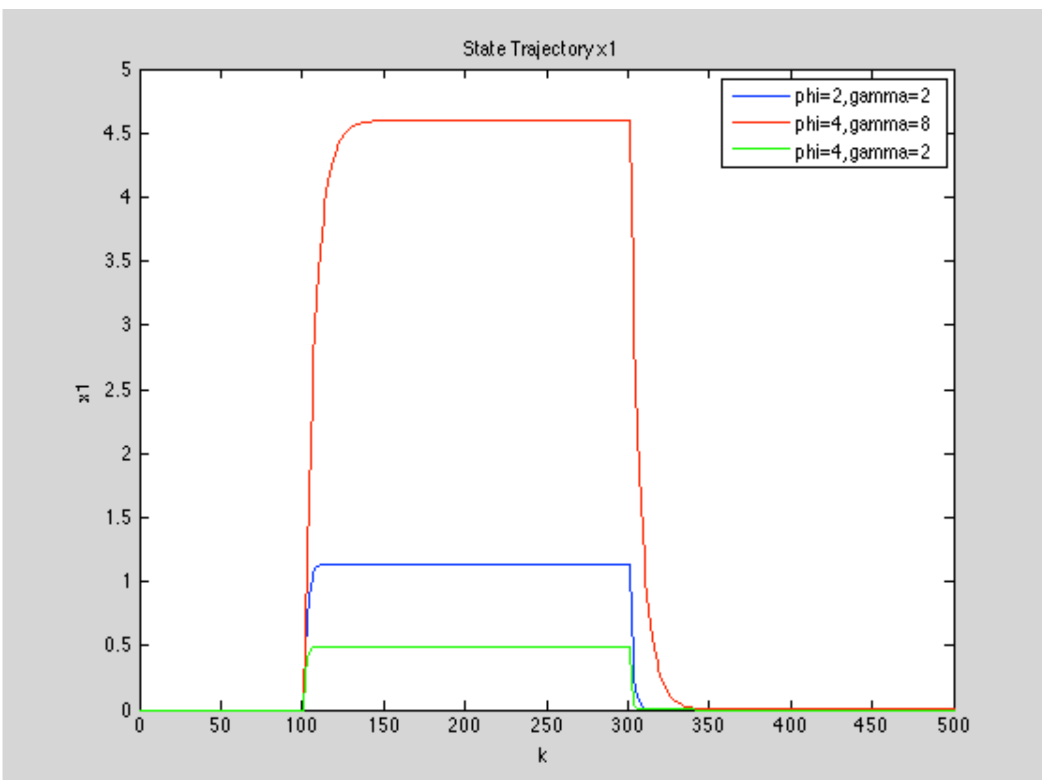

Figure 3.1. 25 System 1-III State Trajectories

To conclude the analysis of the first order case studies, the input, output and state trajectories for System 1-III were shown in Figure 3.1.22-3.1.25. It is seen in the output, input, and state trajectories for each System 1-I, 1-II and 1-III that similar patterns in the trends resulted.

3.2: Second-Order Case Study

The second order case study will also look at three discrete-time systems with varying degrees of stability; System 2-I, System 2-II , and System 2-III.

3.2.1: System Development

The discrete-time canonical system model and output equation for the second-order case studies is shown in (3.2.1-3.2.2). The disturbance model w(k+1) is also shown (3.2.3).

$$x(k+1) = \overline{A}x(k) + \overline{B}u(k) + \overline{F}w(k) \tag{3.2.1}$$

$$y(k) = \overline{C}x(k) + \overline{G}w(k) \tag{3.2.2}$$

$$w(k+1) = \overline{E}w(k) + \sigma(k) \tag{3.2.3}$$

$$\overline{A} = \begin{bmatrix} 1 & 0 \\ -\alpha_1 & -\alpha_2 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \overline{F} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$\overline{C} = [1 \quad 0], \quad \overline{G} = g_o, \quad \overline{E} = 1$$

The coefficients for the system and disturbance models are shown in Table 3 where λ represents the eigenvalues of the system.

Coefficients	System 2-I	System 2-II	System 2-III
α_1	-0.35	-0.55	-0.65
α_2	1.2	1.6	1.8
λ_1	0.5	0.5	0.5
λ_2	0.7	1.1	1.3
g_0	0.064	0.064	0.064

Table 3: Second-Order System Coefficients

The first example is System 2-I, and has eigenvalues within the unit circle as seen in the table. The second eigenvalue of System 2-II and System 2-III, move farther outside of the unit circle. Figure 3.2.1 shows the location of the eigenvalues of each of the system in the second-order case study with respect to the unit circle.

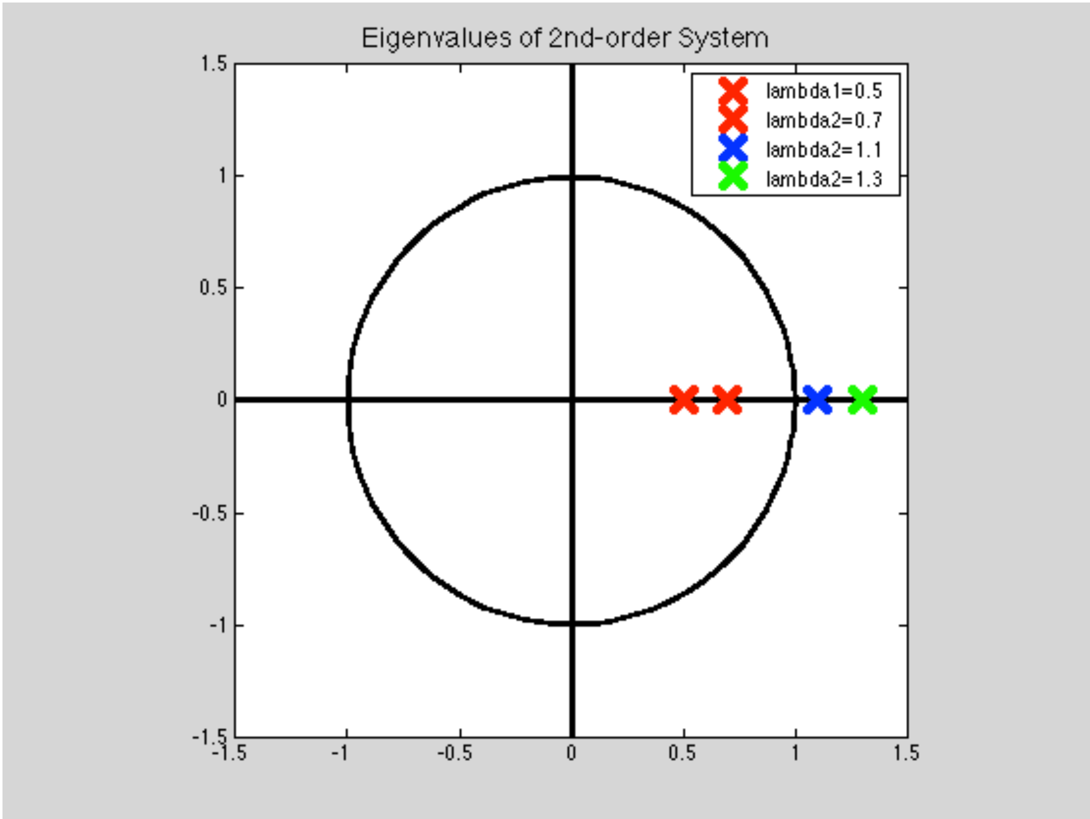

Figure 3.2. 1 System 2-I, 2-II, and 2-III Open-loop Eigenvalues

To create the composite system and design the controller, the pseudo-output z(k) is created, contributing a control term in the output equation. The composite system is then created by augmenting the state update equation and the pseudo-output equation, resulting in a third-order system in terms of ϕ and γ.

$$\begin{bmatrix} x(k+1) \\ z(k+1) \end{bmatrix} = A_p \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} + B_p u(k) + F_p w(k) \tag{3.2.4}$$

$$A_p = \begin{bmatrix} \overline{A} & 0 \\ \phi \overline{C}\overline{A} & 0 \end{bmatrix}, \; B_p = \begin{bmatrix} \overline{B} \\ \phi(\overline{C}\overline{B} + \gamma) \end{bmatrix}, \; F_p = \begin{bmatrix} \overline{F} \\ \phi(\overline{C}\overline{F} + GE) \end{bmatrix}$$

Now that the system matrices have been defined, the controller can be designed.

3.2.2: Controller Design

The controller and disturbance gains are determined in the same manner as for the first order systems using the equations are described in (3.1.6-3.1.7). The application of the controller to the composite system results in the closed-loop system (2.14). The table below has the values for the controller and disturbance gains for each system.

	System I	System II	System III
K_C	$\begin{bmatrix} \frac{0.35}{1+\gamma^2} & \frac{-1.2+\phi\gamma}{1+\gamma^2} & 0 \end{bmatrix}$	$\begin{bmatrix} \frac{0.55}{1+\gamma^2} & \frac{-1.6+\phi\gamma}{1+\gamma^2} & 0 \end{bmatrix}$	$\begin{bmatrix} \frac{0.65}{1+\gamma^2} & \frac{-1.8+\phi\gamma}{1+\gamma^2} & 0 \end{bmatrix}$
K_D	$\frac{1+0.064\phi\gamma}{1+\gamma^2}$	$\frac{1+0.064\phi\gamma}{1+\gamma^2}$	$\frac{1+0.064\phi\gamma}{1+\gamma^2}$

Table 4: Second-Order System Gains

The closed-loop matrix coefficients A_{CL} and F_{CL} are now calculated and used to analyze the stability and disturbance attenuation of the systems.

3.2.3: Analysis

The analysis of the second-order case studies will follow the procedure used for the first-order systems, investigating each system's closed-loop stability, disturbance attenuation, and cost function analysis for various values of ϕ and γ.

This analysis will determine the ideal range of values satisfying the control objectives. Each analysis is performed for values of ϕ and γ from 0 to 20.

System 2-I Evaluation

Considering the stable System 2-I first, the maximum eigenvalues of the closed-loop system are calculated and evaluated for the specified range of ϕ and γ.

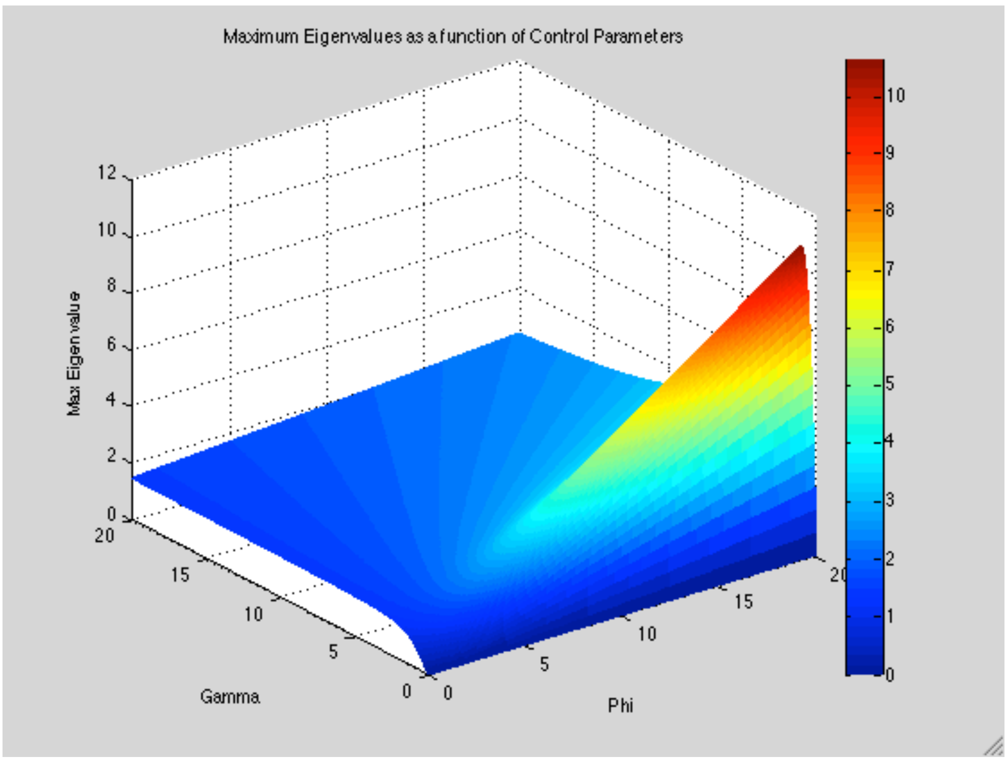

Figure 3.2. 2 System 2-I Maximum Eigenvalues versus ϕ and γ

Figure 3.2.2 shows the result of the stability analysis. In order for the closed-loop system to be stable, its eigenvalues must have a magnitude of less than one to remain within the unit circle. The figure shows that in order for the eigenvalues to remain within the unit circle, values of γ less than 1.35 for all ϕ must be used.

Next, the disturbance accommodation analysis follows the stability investigations. Since the disturbance Grammian G_D is a scalar value due to the disturbance signal being of step-type waveform, it is plotted versus ϕ and γ. In this way, the disturbance attenuation is seen versus the corresponding parameter values.

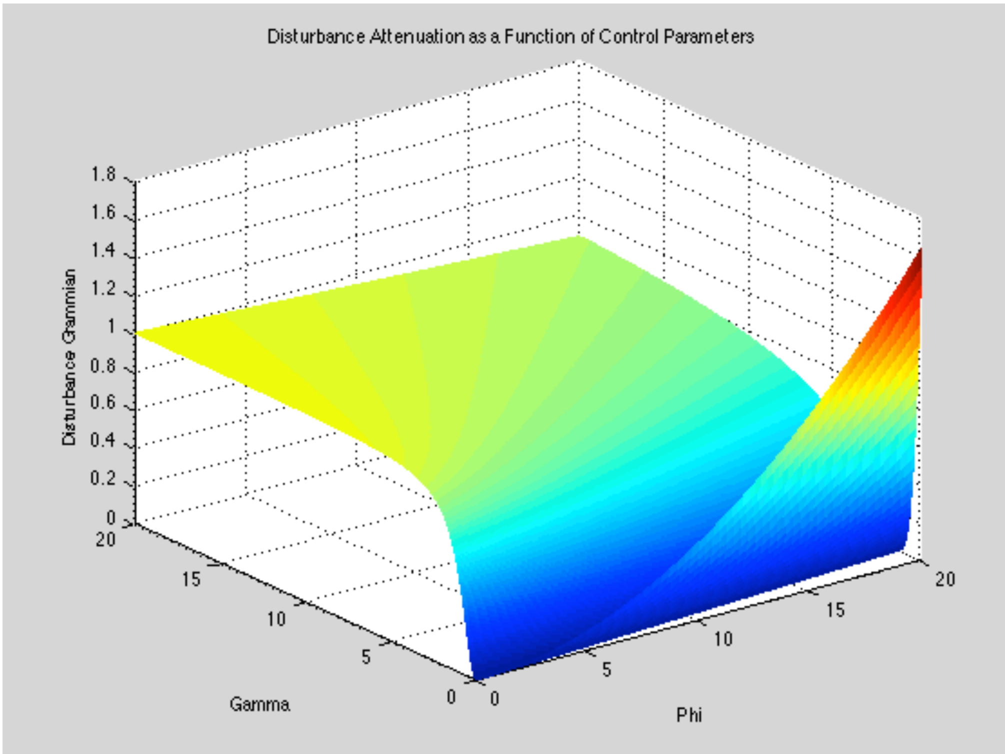

Figure 3.2. 3 System 2-I: Disturbance Grammian versus ϕ and γ

As seen in the figure, there is a large range of acceptable values of ϕ and γ resulting in disturbance attenuation, although for large values of ϕ and small values of γ, the disturbance will not be attenuated. Combining the results of the stability analysis and disturbance accommodation analysis, Figure 3.2.4 shows the acceptable region of phi and gamma needed to achieve both closed-loop system stability and disturbance attenuation.

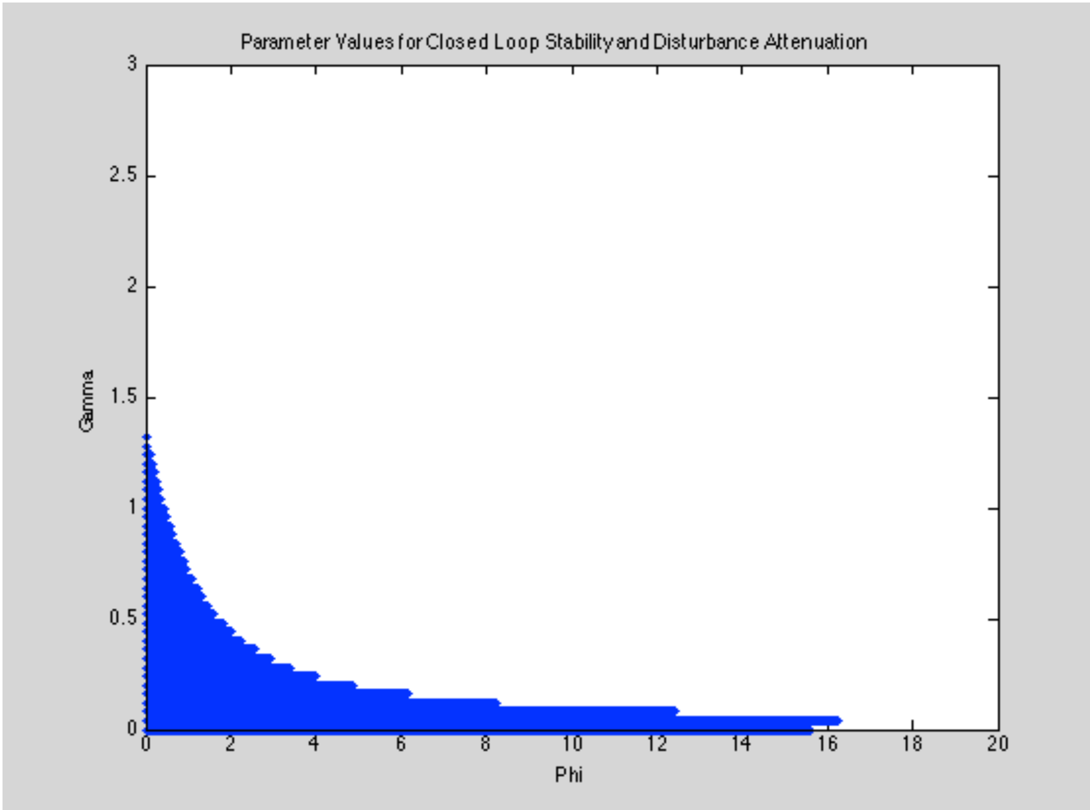

Figure 3.2. 4 Region of Closed-loop Stability and Disturbance Accommodation for System 2-I

The acceptable region of ϕ and γ starts at a value of γ=1.3 when ϕ=0, and decreases as ϕ increases. Figure 3.2.5 shows the cost function analysis for System 2-I. Once again, this analysis was performed by investigating the cost function versus the pseudo-output weighting parameters ϕ and γ.

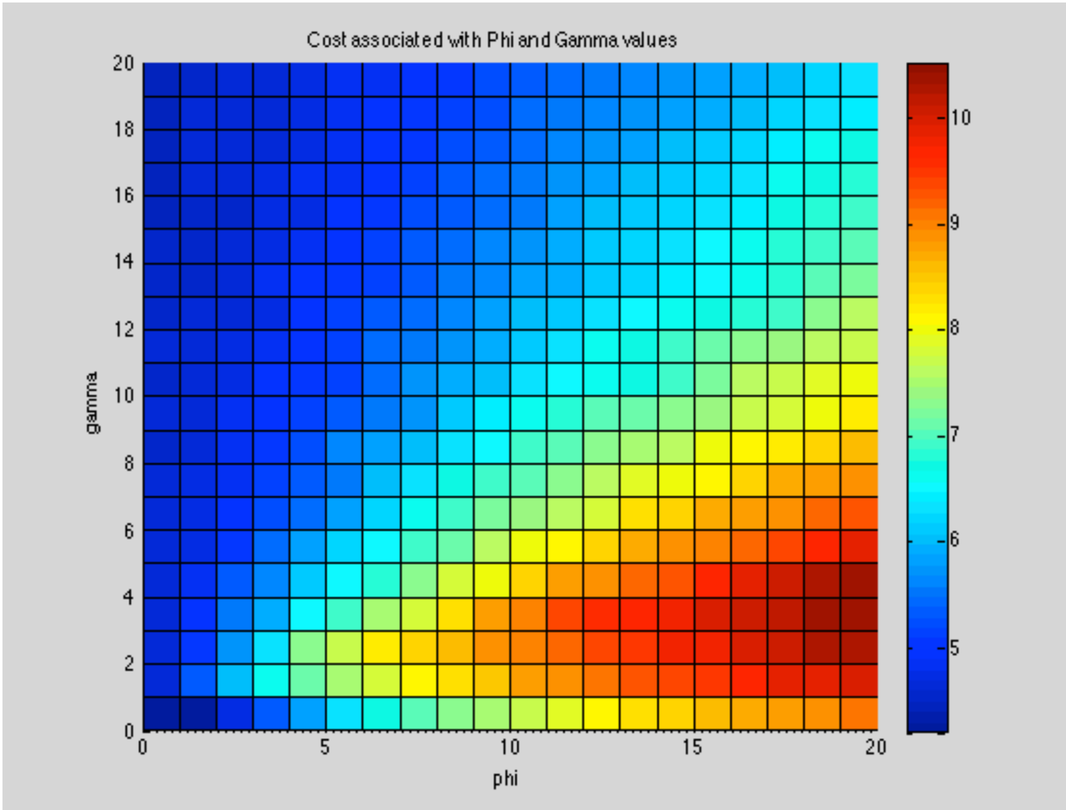

Figure 3.2. 5 Cost Analysis for System 2-I

The cost analysis Figure 3.2.5 shows the cost associated with using any of the particular simulated values of ϕ and γ from 0 to 20. This figure is to be used in conjunction with Figure 3.2.4 to determine the most appropriate values of ϕ and γ to achieve closed-loop stability, disturbance accommodation with the lowest cost. When it comes to closed-loop stability and disturbance accommodation, Figure 3.2.5 is not relevant. The state trajectories for System 2-I are show below. The points were chosen to be able to investigate the response of the system as ϕ and γ increases in value.

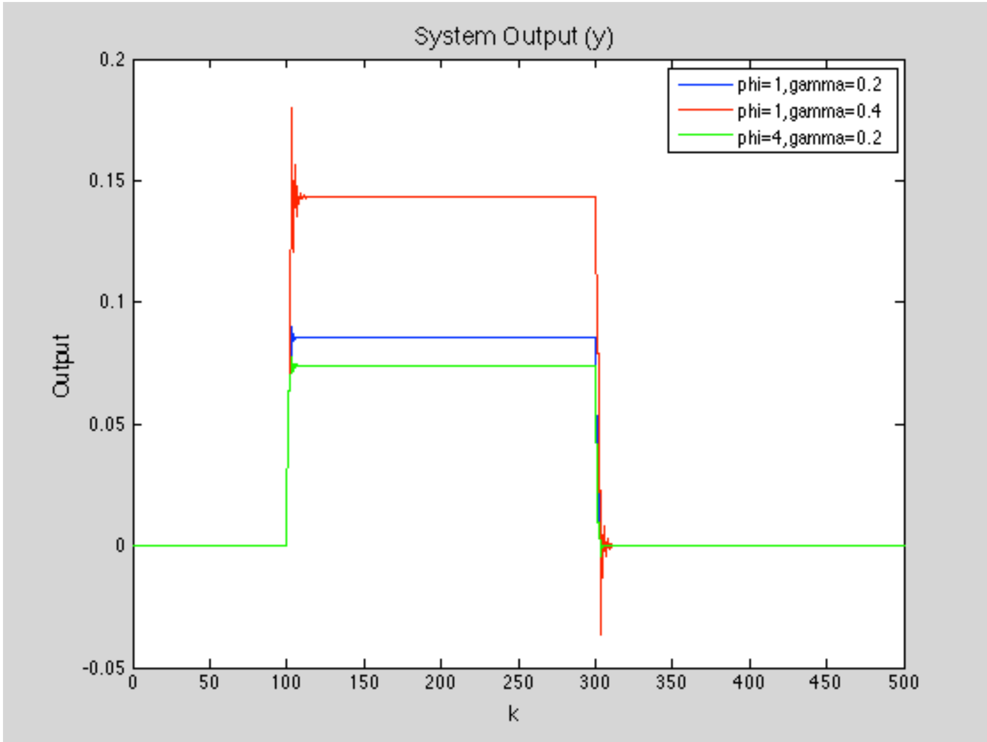

Figure 3.2. 6 System 2-I Output

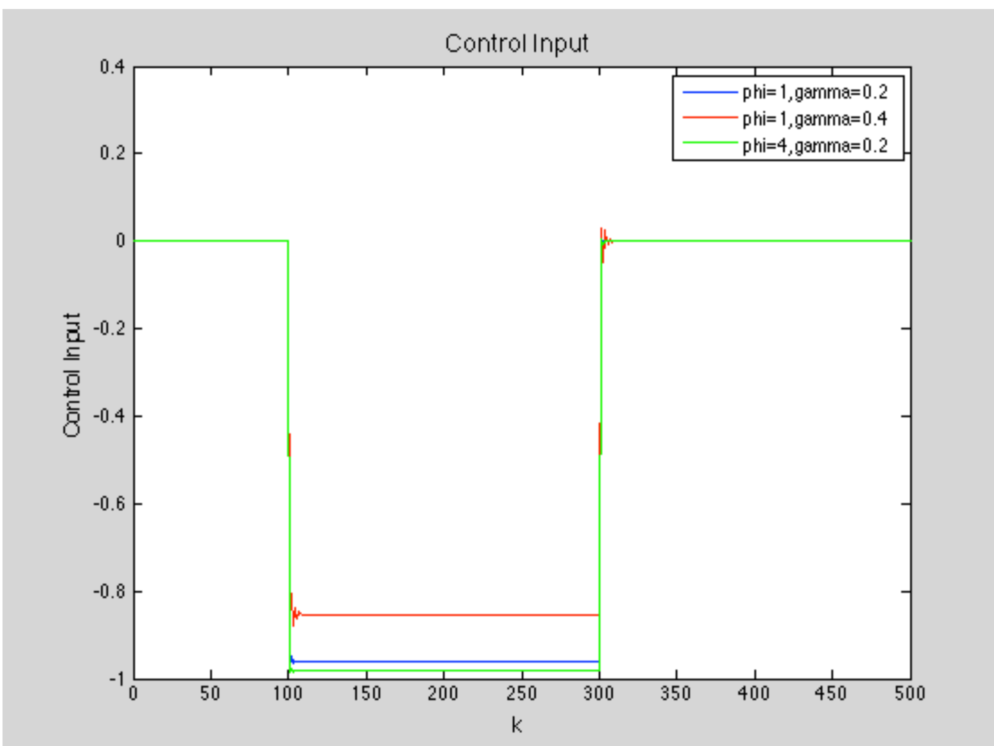

Figure 3.2. 7: System 2-I Control Input

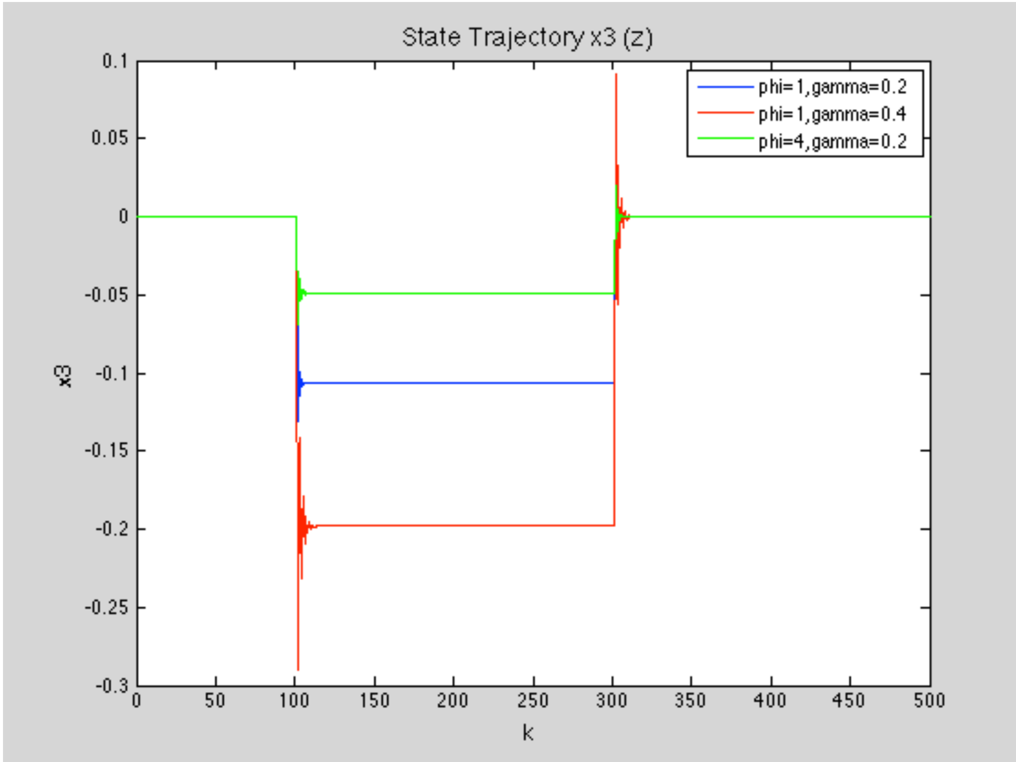

Figure 3.2. 8 System 2-I State Trajectories

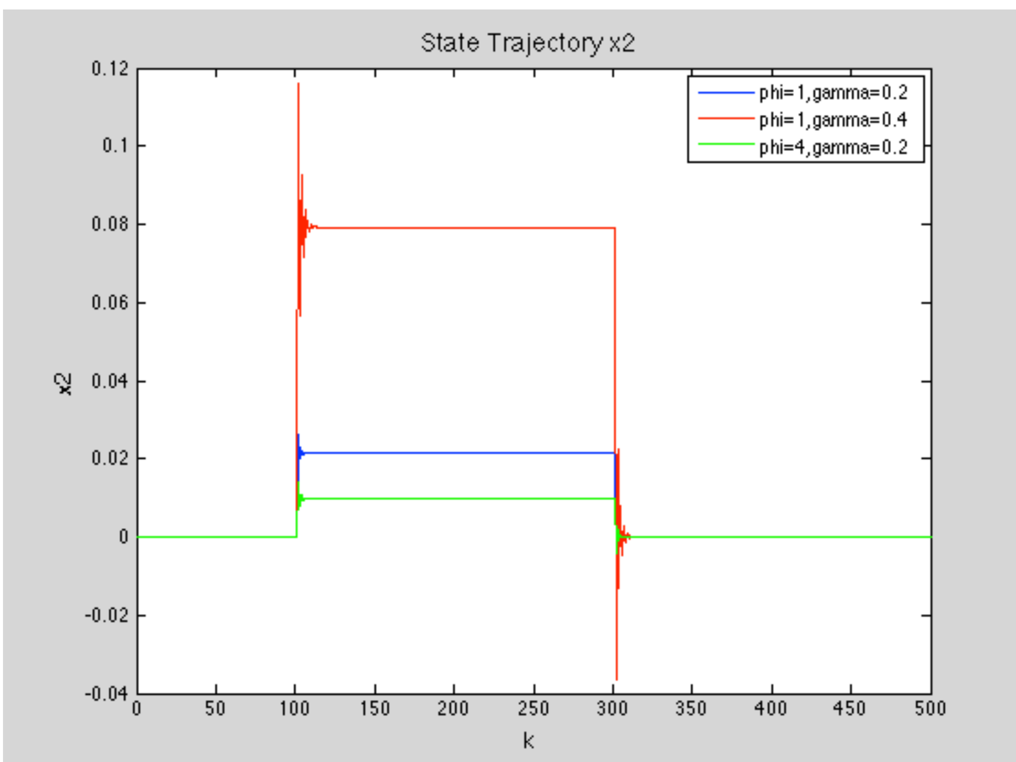

Figure 3.2. 9 System 2-I State Trajectories

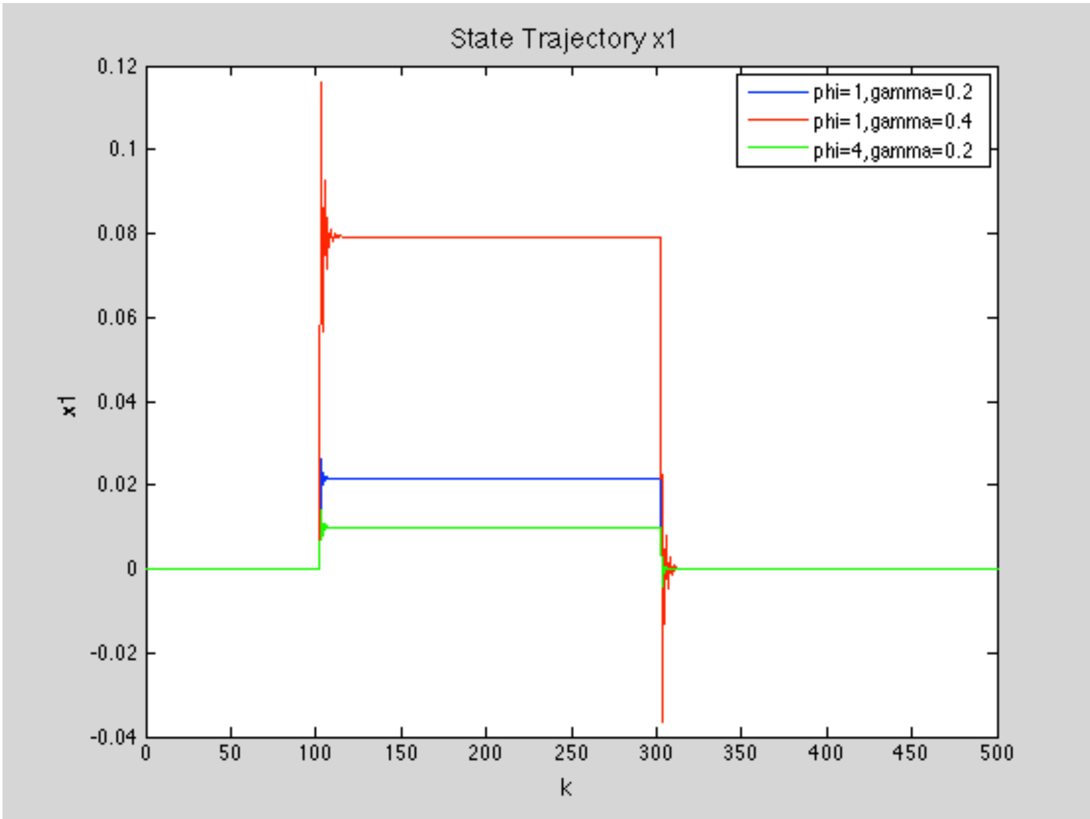

Figure 3.2. 10 System 2-I State Trajectories

Once again it can be observed that the magnitude of the output, input, and respective states depend on the selection of ϕ and γ within the region of acceptable values.

System 2-II Evaluation

The second system in the second-order case studies is System 2-II, whose second eigenvalue is located just outside of the unit circle. The eigenvalues of the closed-loop system versus ϕ and γ, along with the disturbance Grammian ϕ and γ are shown in the following figures.

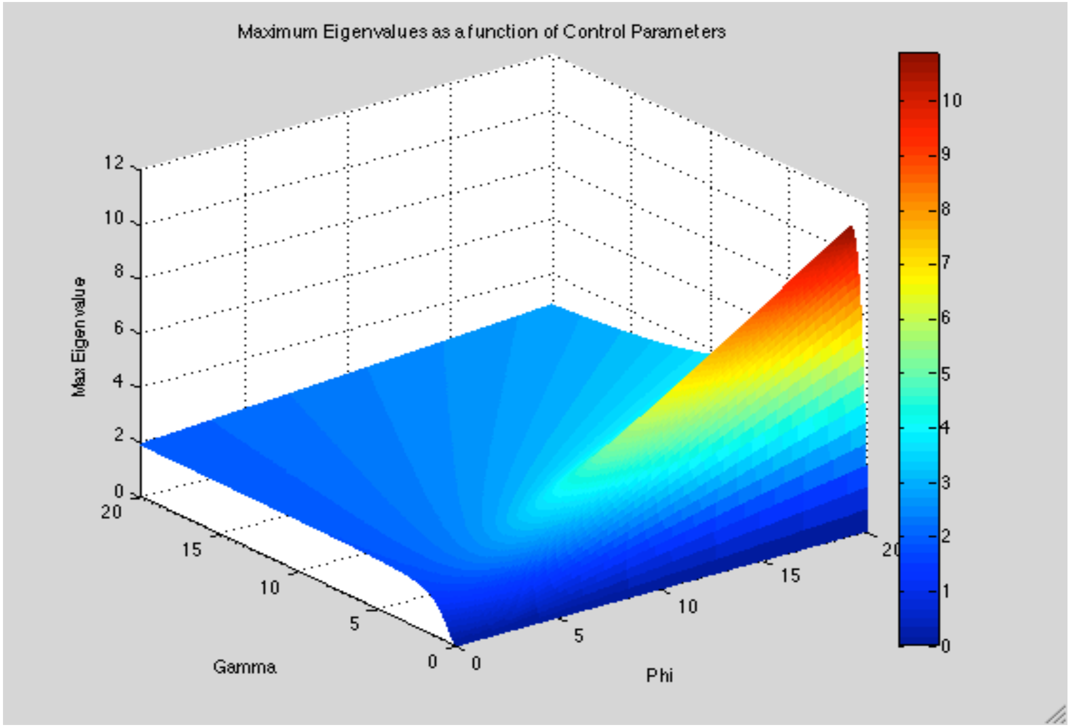

Figure 3.2. 11 System 2-II Max Eigenvalues versus ϕ and γ

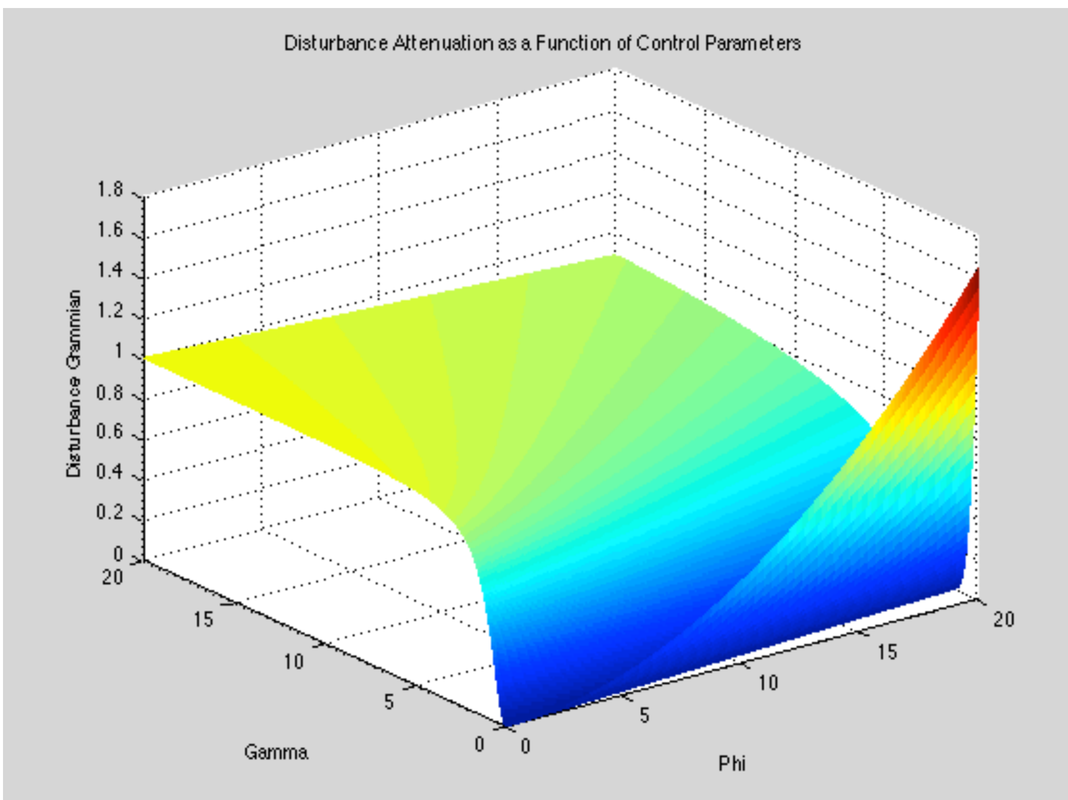

Figure 3.2. 12 System 2-II Disturbance Grammian versus ϕ and γ

To better analyze the acceptable region of ϕ and γ to result in a stable closed-loop system and disturbance accommodation, Figures 3.2.11 and 3.2.12 result in the acceptable region of ϕ and γ seen in Figure 3.2.13.

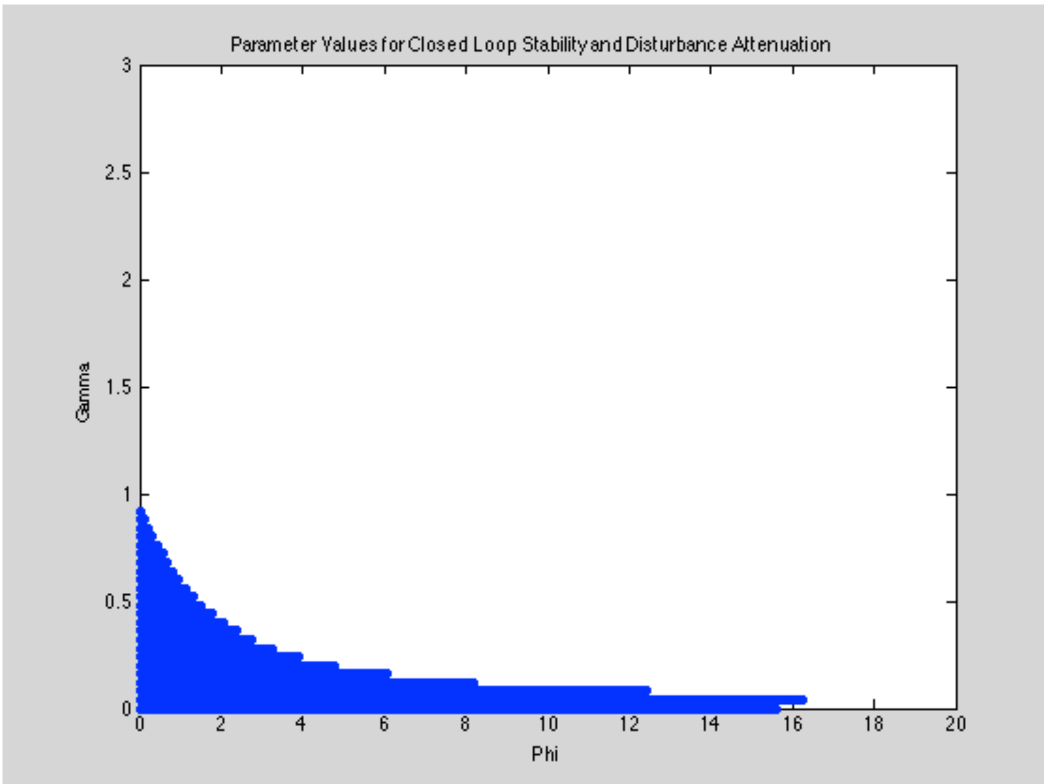

Figure 3.2. 13: Region of Closed-loop Stability and Disturbance Accommodation for System 2-II

Comparing this result to that of System 2-I, the shape of the region of acceptable values is noticeably similar. However, the region is smaller as expected with a value of γ just less than 1 for ϕ equal to zero. The next figure describes the cost analysis for System 2-II.

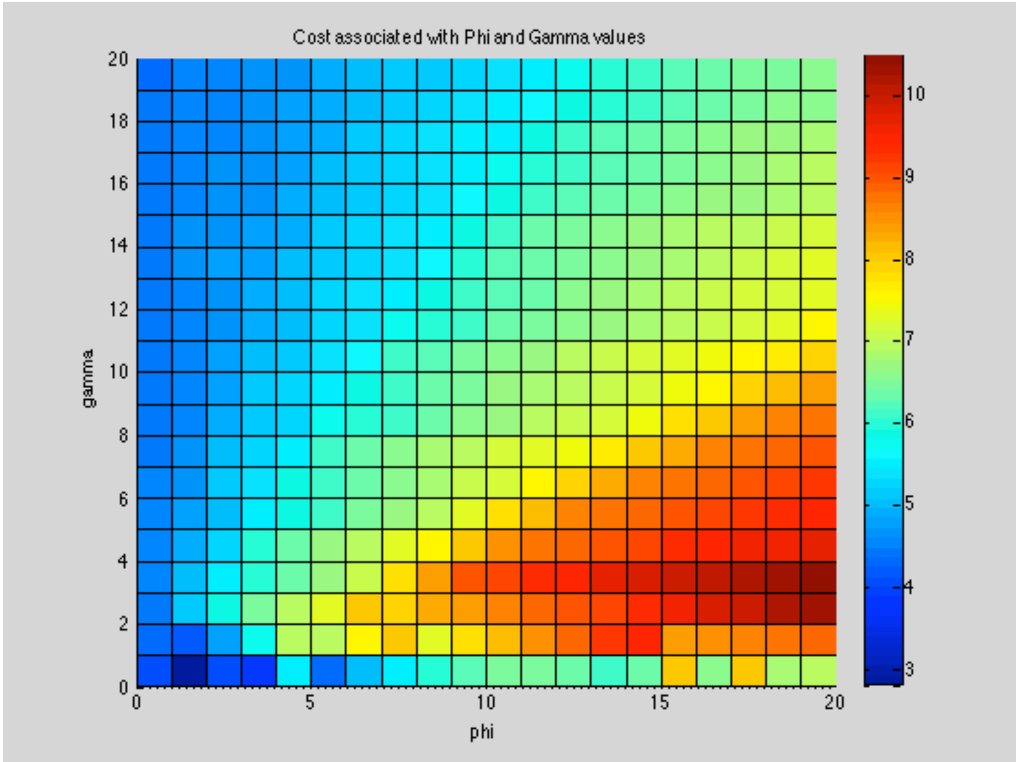

Figure 3.2. 14 Cost Analysis of System 2-II

The cost analysis Figure 3.2.14 shows the cost associated with using any of the particular simulated values of ϕ and γ from 0 to 20 for System 2-II. This figure is to be used together with Figure 3.2.13 to determine the most appropriate values of ϕ and γ to achieve closed-loop stability, disturbance accommodation with the lowest cost. Therefore, values to simulate the trajectories were chosen from the acceptable region in Figure 3.2.13, considering the cost analysis from Figure 3.2.14. The same values of ϕ and γ to simulate the trajectories as in System 2-I, and the figures below show the output response, control input, and state variables of the system.

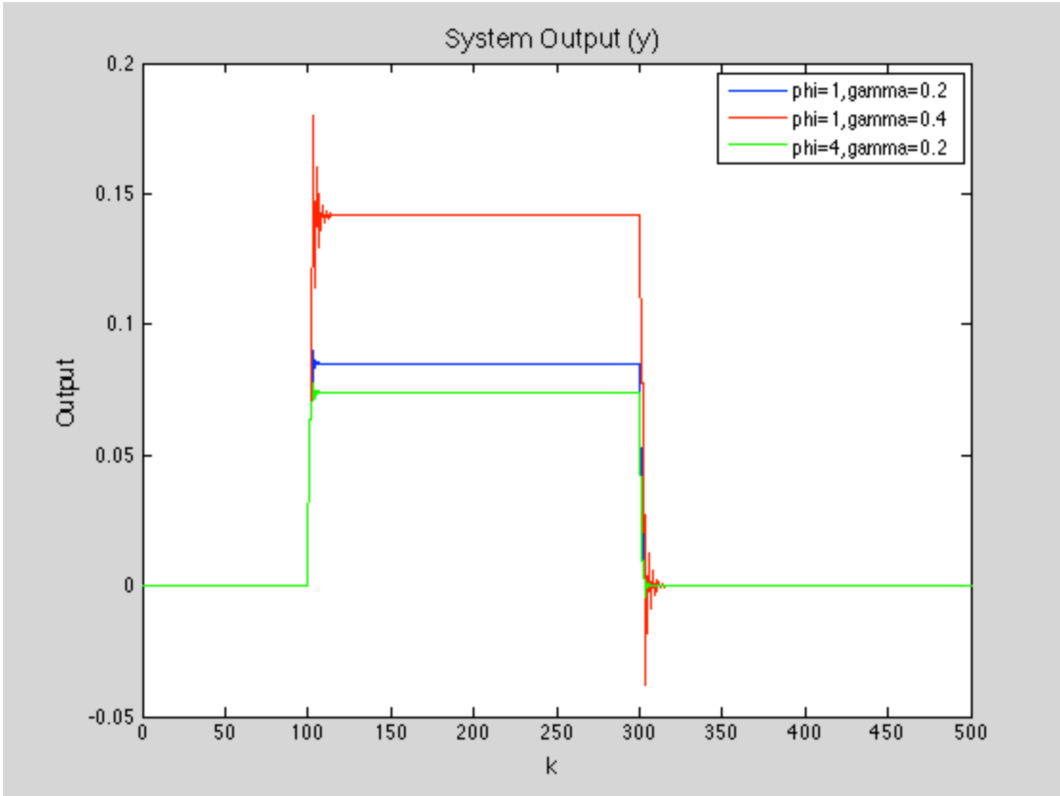

Figure 3.2. 15 System 2-II Output

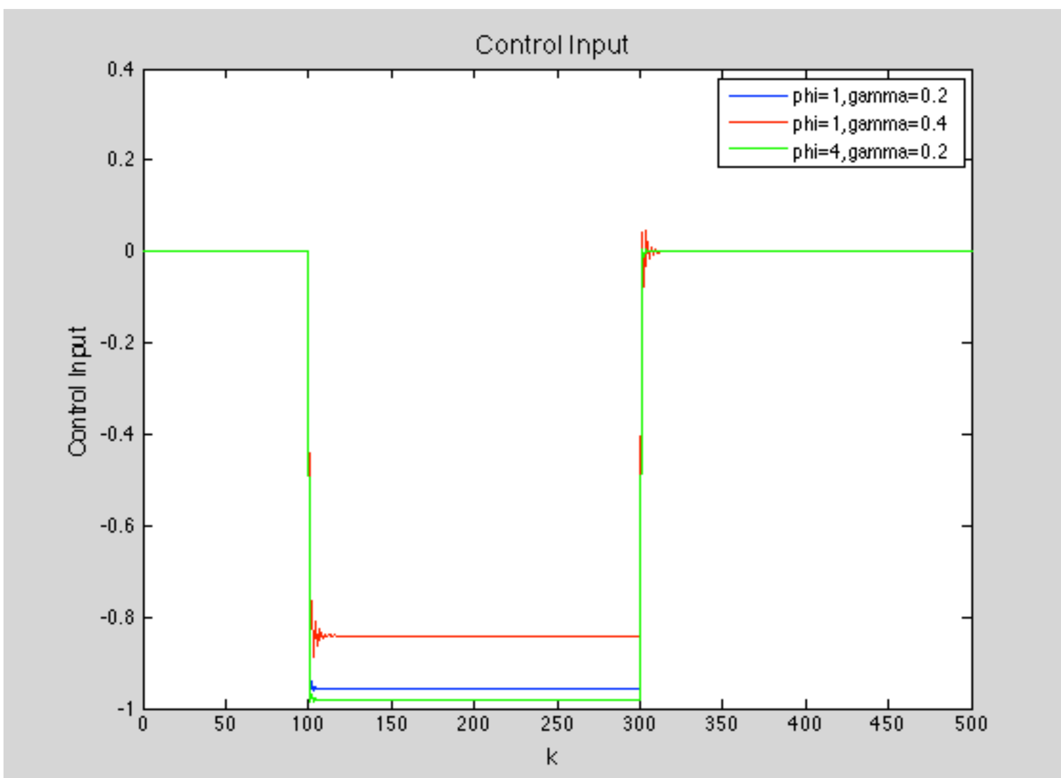

Figure 3.2. 16 System 2-II Control Input

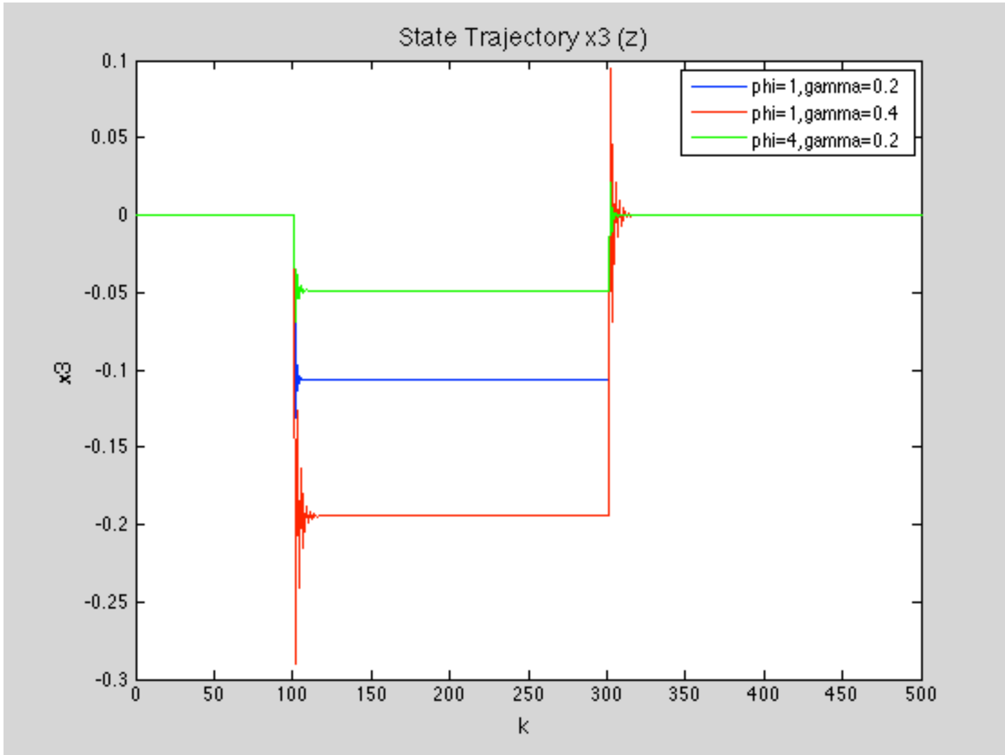

Figure 3.2. 17 System 2-II State Trajectories

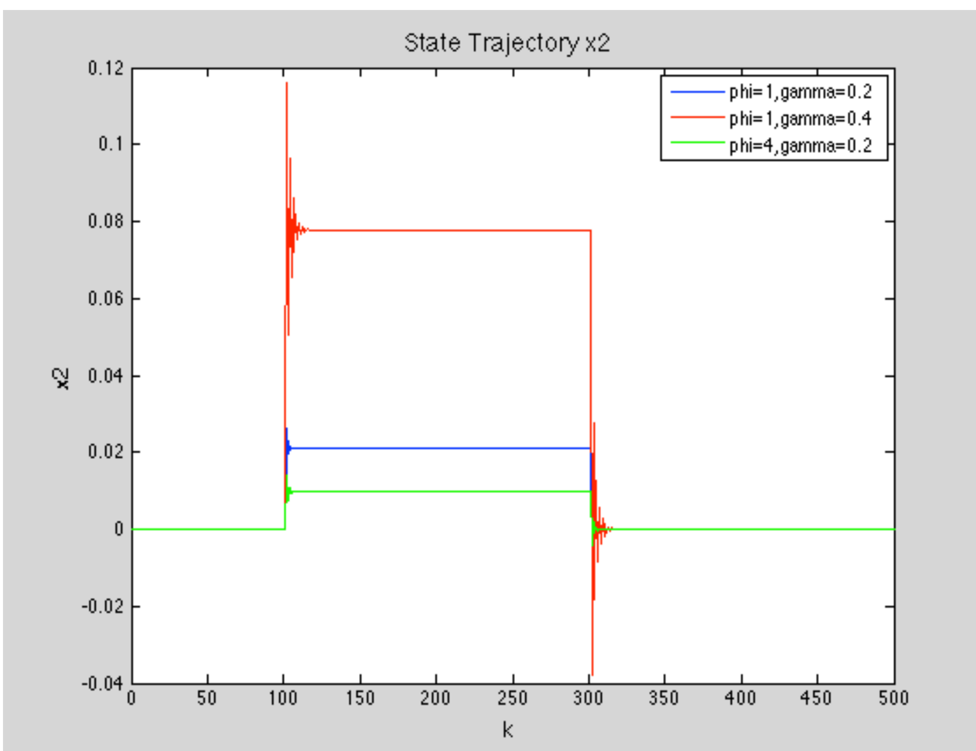

Figure 3.2. 18 System 2-II State Trajectories

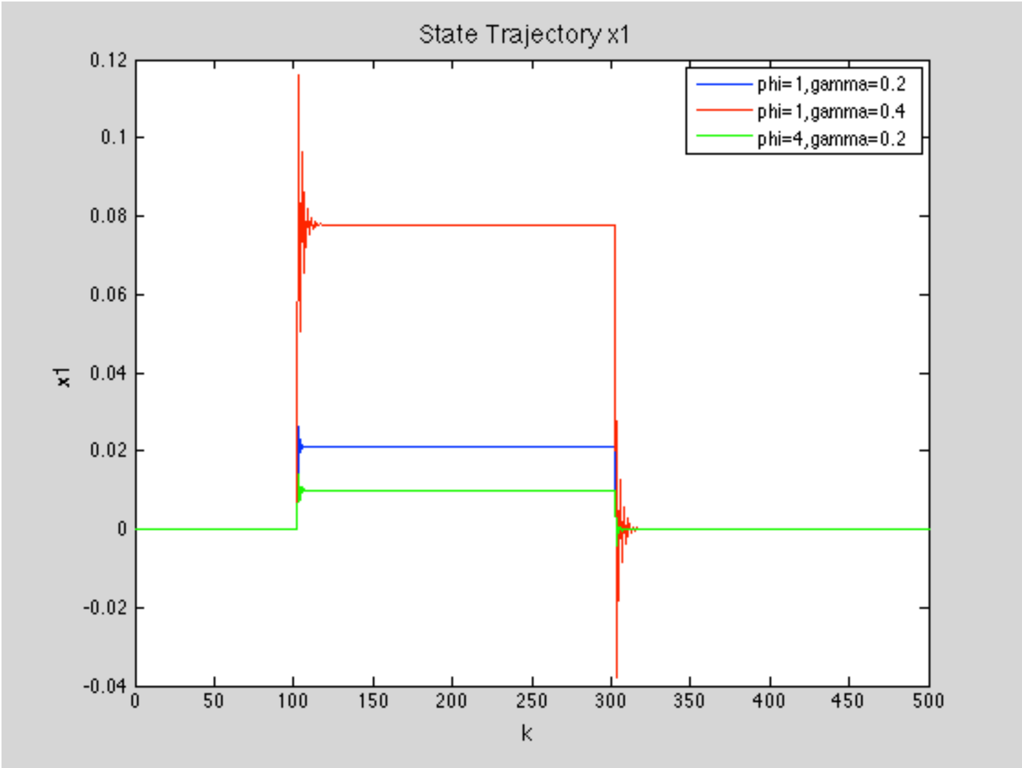

Figure 3.2. 19 System 2-II State Trajectories

The presented state trajectories for System 2-II continue to exhibit similar trends, dependant on ϕ and γ. The output seen in Figure 3.2.15 shows that as ϕ is increased and γ kept constant, the magnitude of y decreases. While ϕ is kept constant and γ increased, the magnitude of y increases. When looking at the input in Figure 3.2.16, the magnitude of u increases as ϕ is increased and γ is kept constant. The magnitude of u decreases when γ is increased and ϕ is kept constant. Looking at the state trajectories in 3.2.17-3.2.19 shows that when ϕ is increased and γ is kept constant, the magnitude of the state variables decreases. On the other hand, when γ is increased and ϕ is kept constant, the magnitude of the state variables increases.

System 2-III Evaluation

To conclude the analysis of the second-order case studies, System 2-III will now be considered. This system is the most unstable of the second-order open-loop systems, with its second eigenvalue farthest outside of the unit circle. The following figures are the result from the stability analysis, disturbance accommodation analysis as well as the merging of those results to yield an acceptable region of ϕ and γ.

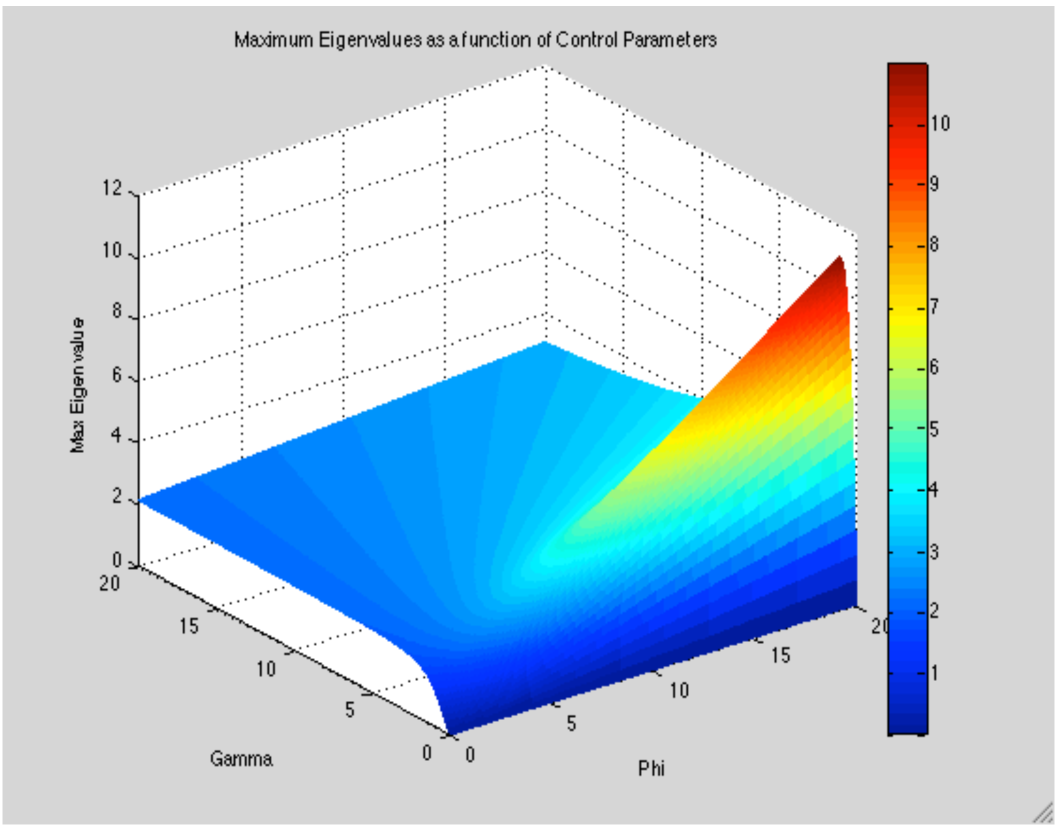

Figure 3.2. 20 Max Eigenvalues of System 2-III versus ϕ and γ

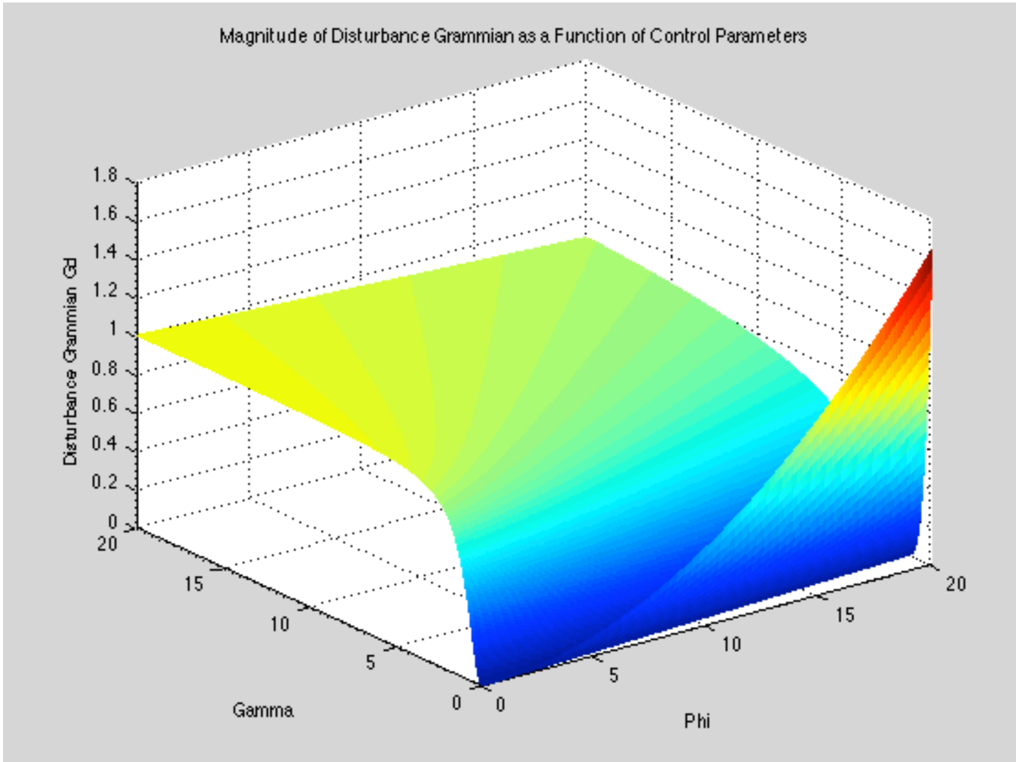

Figure 3.2. 21 System 2-III Disturbance Grammian versus ϕ and γ

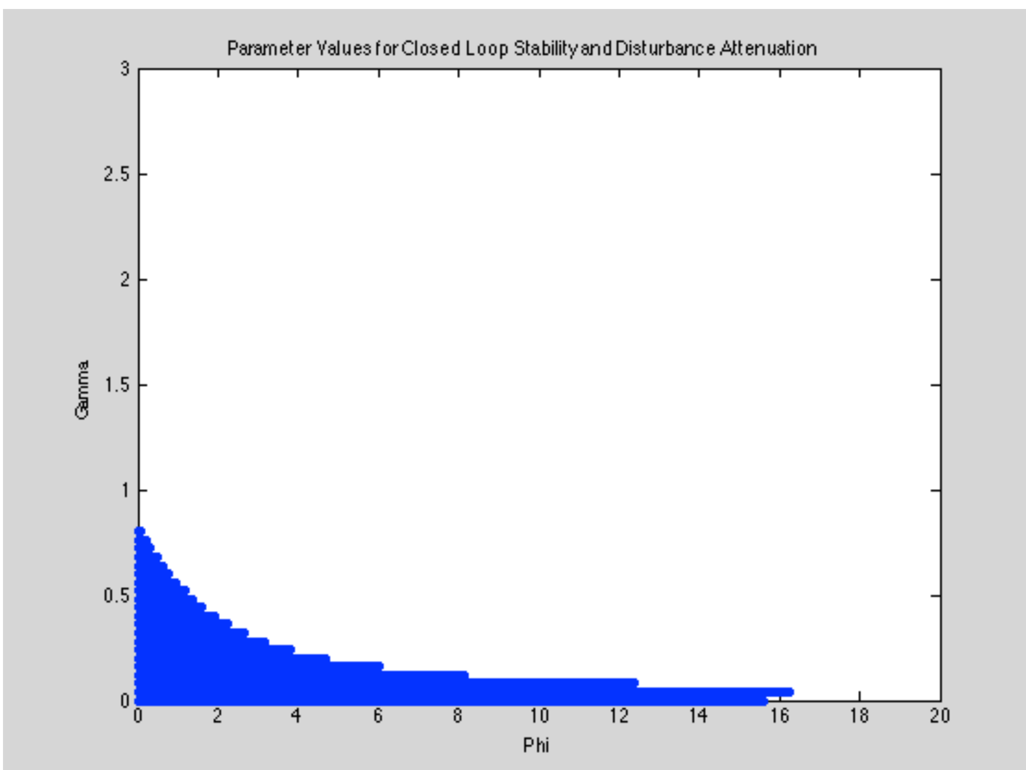

Figure 3.2. 22 Region of Closed-loop Stability and Disturbance Accommodation for System 2-III

Once again, since the eigenvalues of the open-loop System 2-III are farthest outside the unit circle, it is reasonable that the acceptable region of ϕ and γ values is the smallest out of all three systems. This can also be seen by the eigenvalues less than one in Figure 3.2.20 and the magnitude of the disturbance Grammian less than one in Figure 3.2.21.

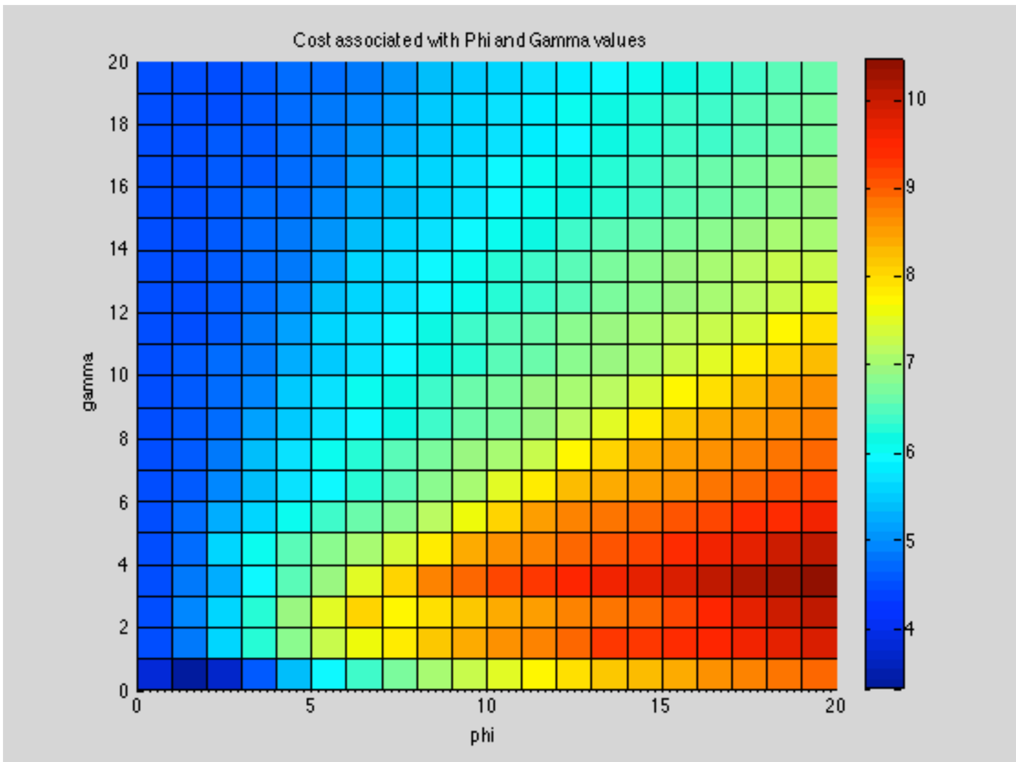

Figure 3.2. 23 Cost Analysis System 2-III

The cost function analysis plot in Figure 3.2.23 shows that the regions of lowest cost associated with using any of the particular simulated values of ϕ and γ from 0 to 20 for System 2-III. This figure is to be used along with Figure 3.2.22 to determine the most appropriate values of ϕ and γ to achieve closed-loop stability, disturbance accommodation with the lowest possible cost. Once again, the same three values of ϕ and γ that result in a stable closed loop system and disturbance accommodation were used from Figure 3.2.22 with the corresponding costs from 3.2.23. The subsequent figures present the trajectories of System 2-III for the selected values of ϕ and γ.

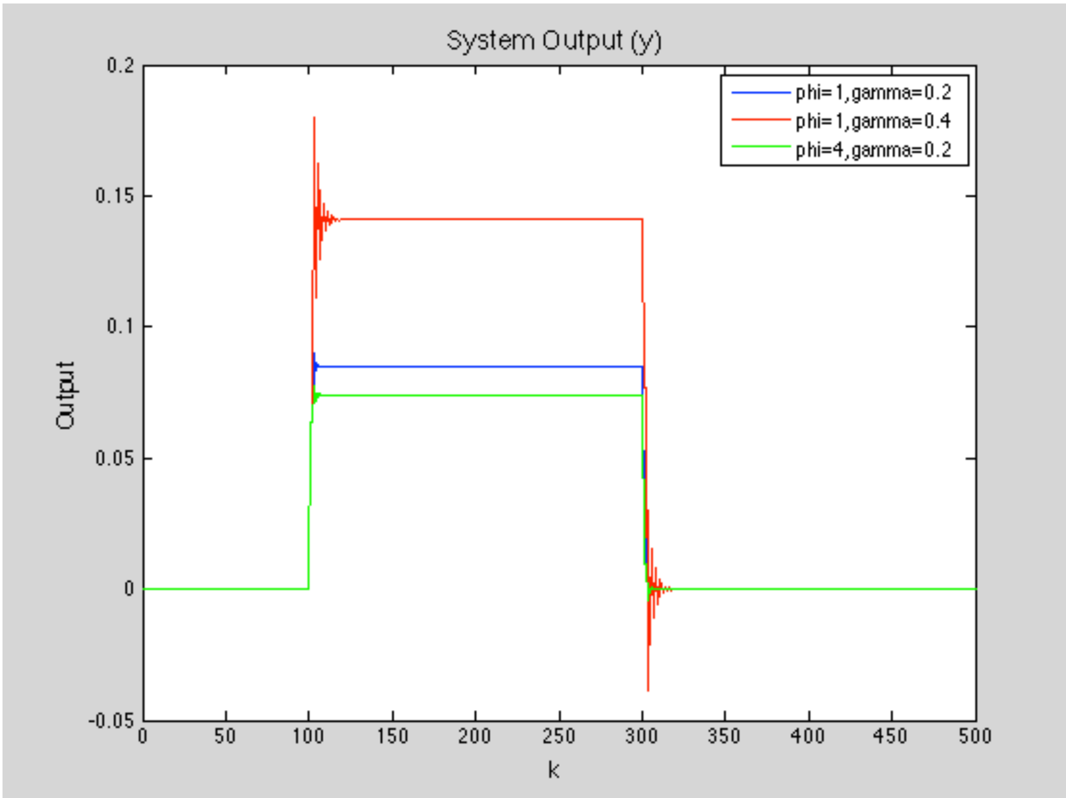

Figure 3.2. 24 System 2-III Output

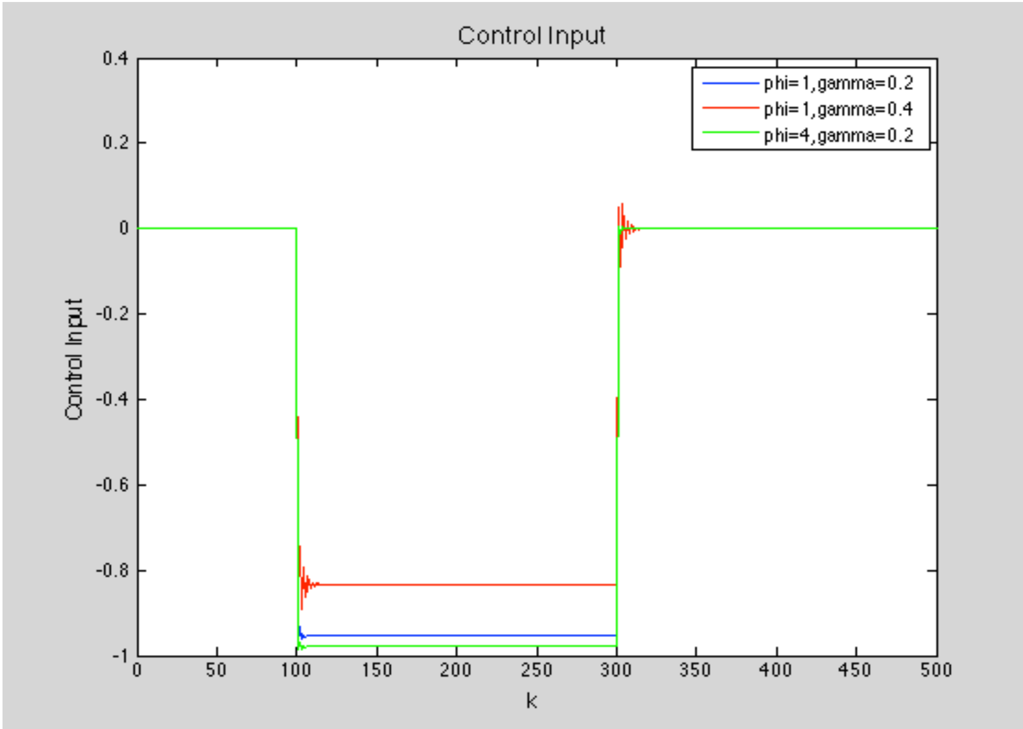

Figure 3.2. 25 System 2-III Control Input

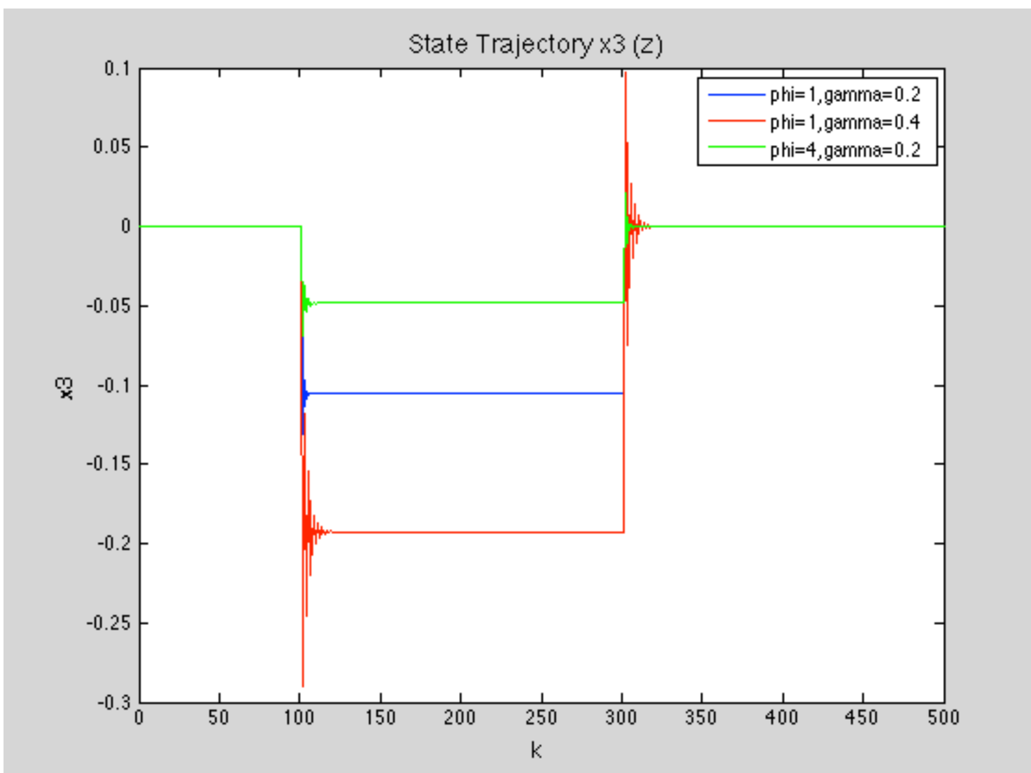

Figure 3.2. 26 System 2-III State Trajectories

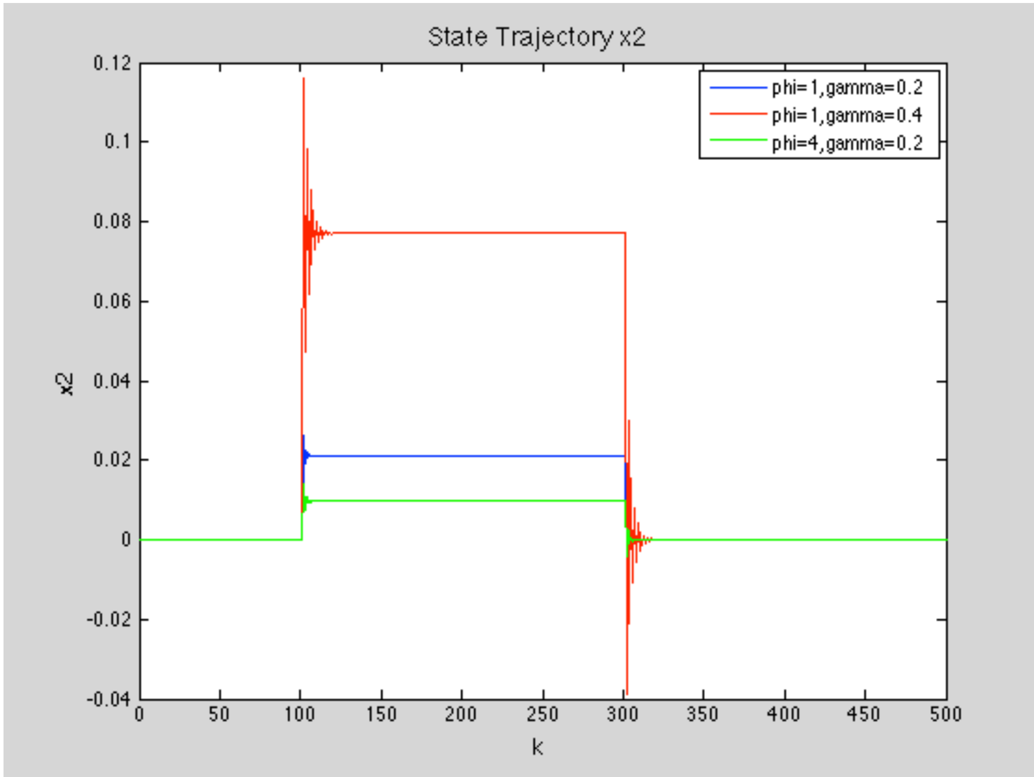

Figure 3.2. 27 System 2-III State Trajectories

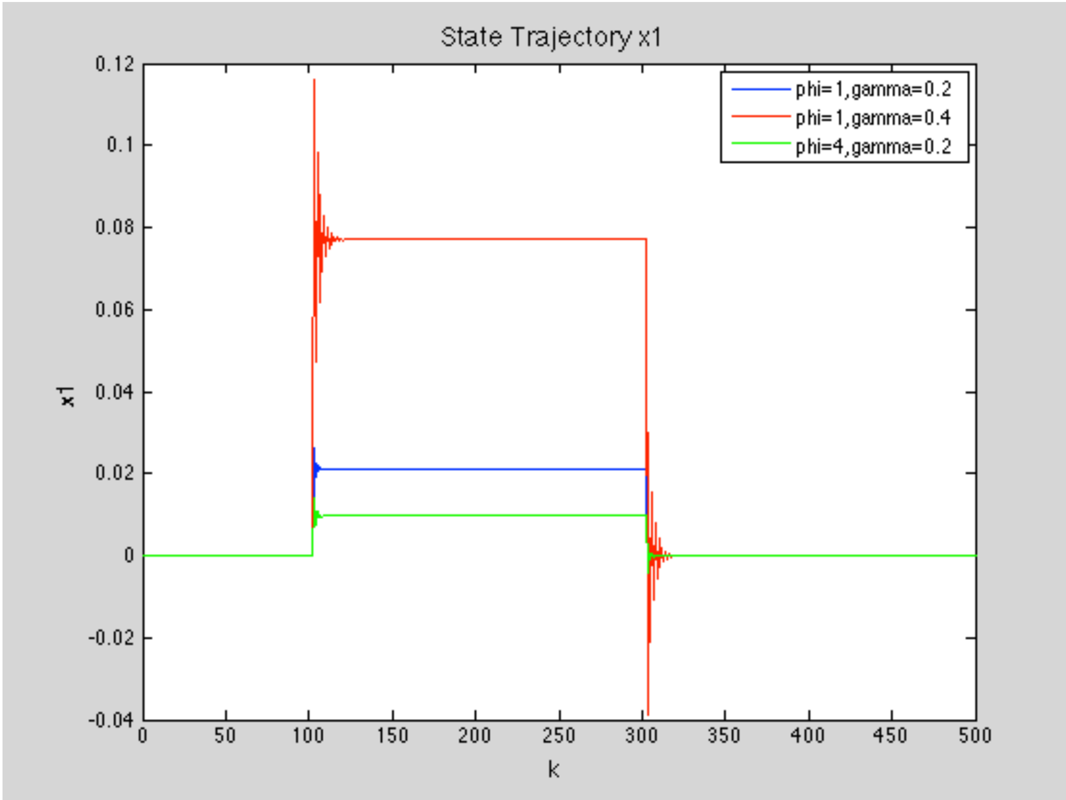

Figure 3.2. 28 System 2-III State Trajectories

The output seen in Figure 3.2.24 shows that as ϕ is increased and γ kept constant, the magnitude of *y* decreases. While ϕ is kept constant and γ increased, the magnitude of *y* increases. When looking at the input in Figure 3.2.25, the magnitude of *u* increases as ϕ is increased and γ is kept constant. The magnitude of *u* decreases when γ is increased and ϕ is kept constant. Looking at the state trajectories in 3.2.26-3.2.28 shows that when ϕ is increased and γ is kept constant, the magnitude of the state variables decreases. When γ is increased and ϕ is kept constant, the magnitude of the state variables increases.

To conclude the analysis of the second-order systems, it was seen that the response of the system depends on the values chosen for ϕ and γ. A region of acceptable ϕ and γ values was found to guarantee closed-loop stability and disturbance accommodation, and the cost analysis for each system performed. It can be said that for each of the second-order case study systems, the objectives were met, although there is slightly less flexibility in the selection of ϕ and γ parameter values as the open-loop system becomes less stable.

3.3: Third-Order Case Study

The final case study in the analysis of this work is the third-order systems analysis. The analysis of the third-order stable System 3-I will commence, followed by Systems 3-II and 3-III, decreasing in stability.

3.3.1: System Development

The discrete-time canonical system model and disturbance model for the third-order case studies are shown below.

$$x(k+1) = \overline{A}x(k) + \overline{B}u(k) + \overline{F}w(k) \tag{3.3.1}$$

$$y(k) = \overline{C}x(k) + \overline{G}w(k) \tag{3.3.2}$$

$$w(k+1) = \overline{E}w(k) + \sigma(k) \tag{3.3.3}$$

The matrix coefficients for the models in (3.3.1-3.3.3) are shown, where g_0 is a scalar value.

$$\overline{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_1 & -\alpha_2 & -\alpha_3 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \overline{F} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\overline{C} = [1 \quad 0 \quad 0], \quad \overline{G} = g_o, \quad \overline{E} = 1$$

Considering this model, the coefficients and parameter values for each System 3-I, System 3-II and System 3-III are shown in the table below, where **λ** represents the open-loop system eigenvalues.

Coefficients	System 3-I	System 3-II	System 3-III
α_1	-0.04	-0.055	-0.065
α_2	0.53	0.71	0.83
α_3	-1.4	-1.7	-1.9
λ_1	0.1	0.1	0.1
λ_2	0.5	0.5	0.5
λ_3	0.8	1.1	1.3
g_0	0.064	0.064	0.064

Table 5: Third-order System Coefficients

The location of the eigenvalues of each of the three systems on the unit circle is illustrated in the figure below. System 3-I is stable, with its eigenvalues within the unit circle, while System 3-II and System 3-III are increasingly unstable as their eigenvalues move outside the unit circle.

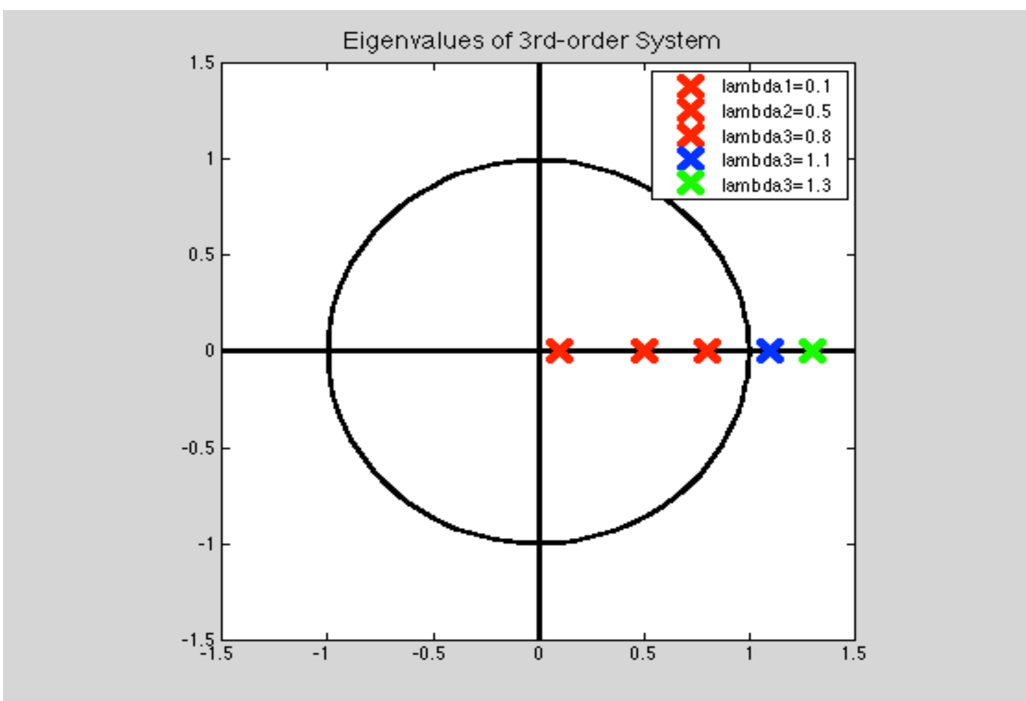

Figure 3.3. 1 System 3-I, 3-II, and 3-III Open-loop Eigenvalues

The design of the composite system follows as for the previous case study analyses. Once again, the state update equation is augmented with the pseudo-output update equation to result in the new system model, now in terms of the pseudo-output weighting parameters, ϕ and γ. The analysis of these systems will determine the appropriate range of values for these parameters to guarantee closed-loop stability, disturbance accommodation and control input minimization.

$$\begin{bmatrix} x(k+1) \\ z(k+1) \end{bmatrix} = A_p \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} + B_p u(k) + F_p w(k) \tag{3.3.4}$$

$$A_p = \begin{bmatrix} \bar{A} & 0 \\ \phi \bar{C}\bar{A} & 0 \end{bmatrix},\ B_p = \begin{bmatrix} \bar{B} \\ \phi(\bar{C}\bar{B} + \gamma) \end{bmatrix},\ F_p = \begin{bmatrix} \bar{F} \\ \phi(\bar{C}\bar{F} + GE) \end{bmatrix}$$

3.3.2: Controller Design

Following the system development, the controller gains and disturbance gains must be determined. Using (3.1.6-3.1.7), the controller gains and disturbance gains are designed and the closed-loop system presented. The table below presents the appropriate gains and coefficient matrices for each system.

	System 3-I	**System 3-II**	**System 3-III**
K_C	$\left[\frac{0.04}{1+\gamma^2} \quad \frac{-0.53+\phi\gamma}{1+\gamma^2} \quad \frac{1.4}{1+\gamma^2} \quad 0\right]$	$\left[\frac{0.04}{1+\gamma^2} \quad \frac{-0.53+\phi\gamma}{1+\gamma^2} \quad \frac{1.4}{1+\gamma^2} \quad 0\right]$	$\left[\frac{0.04}{1+\gamma^2} \quad \frac{-0.53+\phi\gamma}{1+\gamma^2} \quad \frac{1.4}{1+\gamma^2} \quad 0\right]$
K_D	$\frac{1+0.064\phi\gamma}{1+\gamma^2}$	$\frac{1+0.064\phi\gamma}{1+\gamma^2}$	$\frac{1+0.064\phi\gamma}{1+\gamma^2}$

Table 6: Third-Order Controller and Disturbance Gains

3.3.3: Analysis

For the analysis of the third-order case studies, the closed-loop stability, disturbance accommodation, and cost function analysis are all performed as a function of ϕ and γ, the pseudo-output weighting parameters.

System 3-I Evaluation

The stability and disturbance accommodation analyses presented will result in an acceptable region of ϕ and γ values to guarantee stability and disturbance accommodation. Upon performing the cost analysis, specific values of ϕ and γ are chosen to simulate the output, control input, and state trajectories for each system.

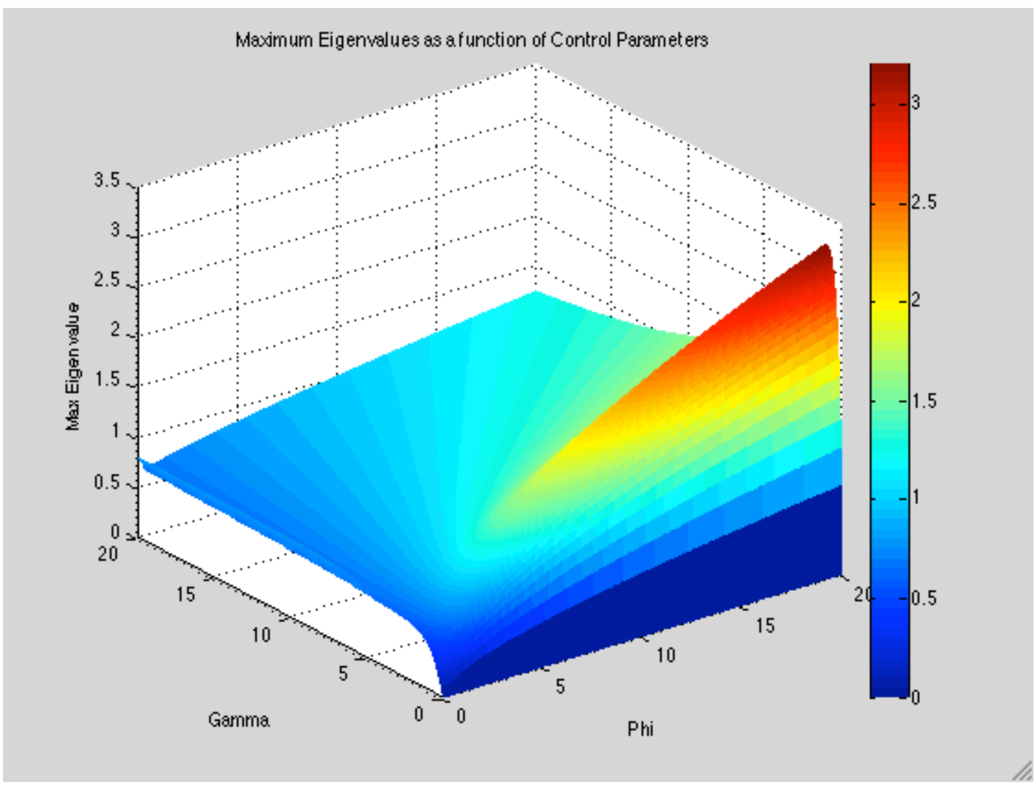

Figure 3.3. 2 System 3-I Max Eigenvalues versus ϕ and γ

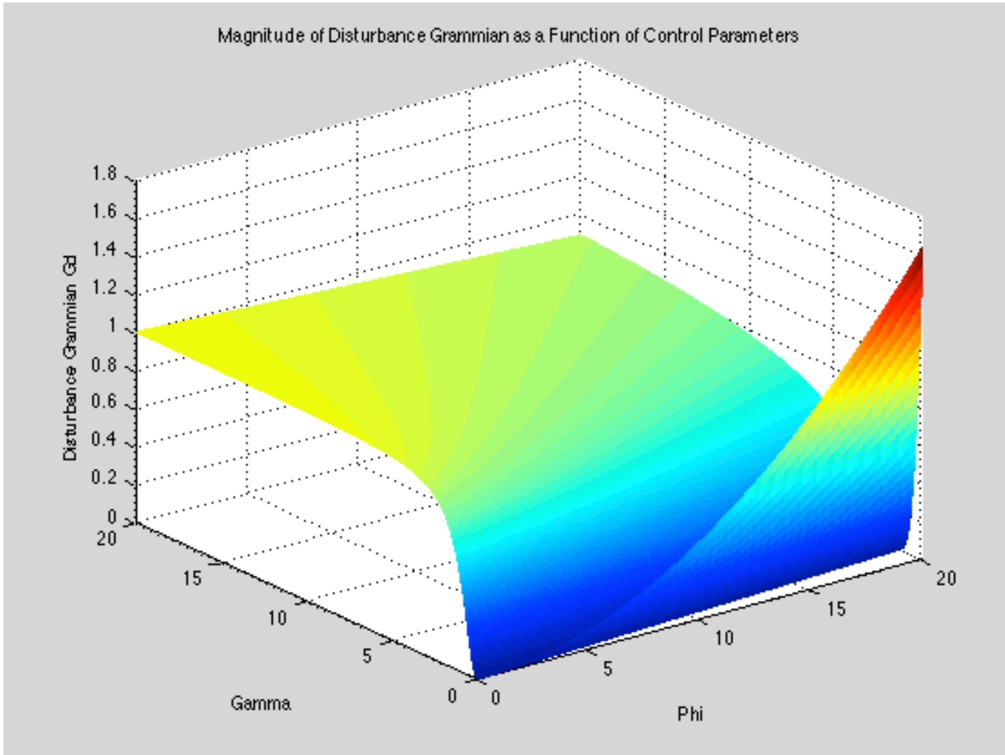

Figure 3.3. 3 System 3-1 Disturbance Grammian versus ϕ and γ

The results of interest from the figure showing the closed-loop eigenvalues and disturbance attenuation are those when the closed-loop eigenvalues are within the unit circle, and when the disturbance Grammian has a magnitude of less than one. Taking this into consideration, the figure below shows the region of ϕ and γ values that will yield a stable closed-loop system while achieving disturbance accommodation.

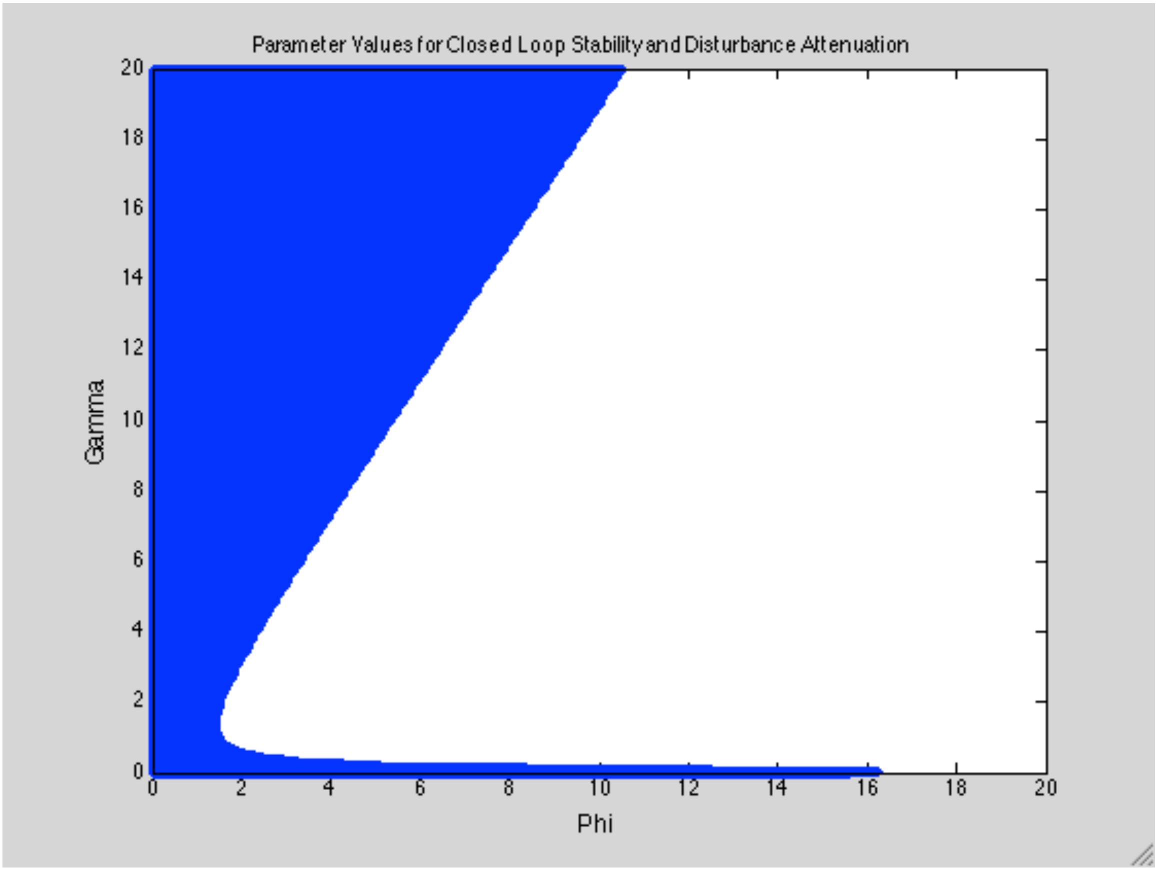

Figure 3.3. 4 Region of Closed-loop Stability and Disturbance Accommodation for System 3-I

After the determination of an acceptable range of ϕ and γ to achieve closed-loop stability and disturbance accommodation, the cost analysis is performed and is shown below.

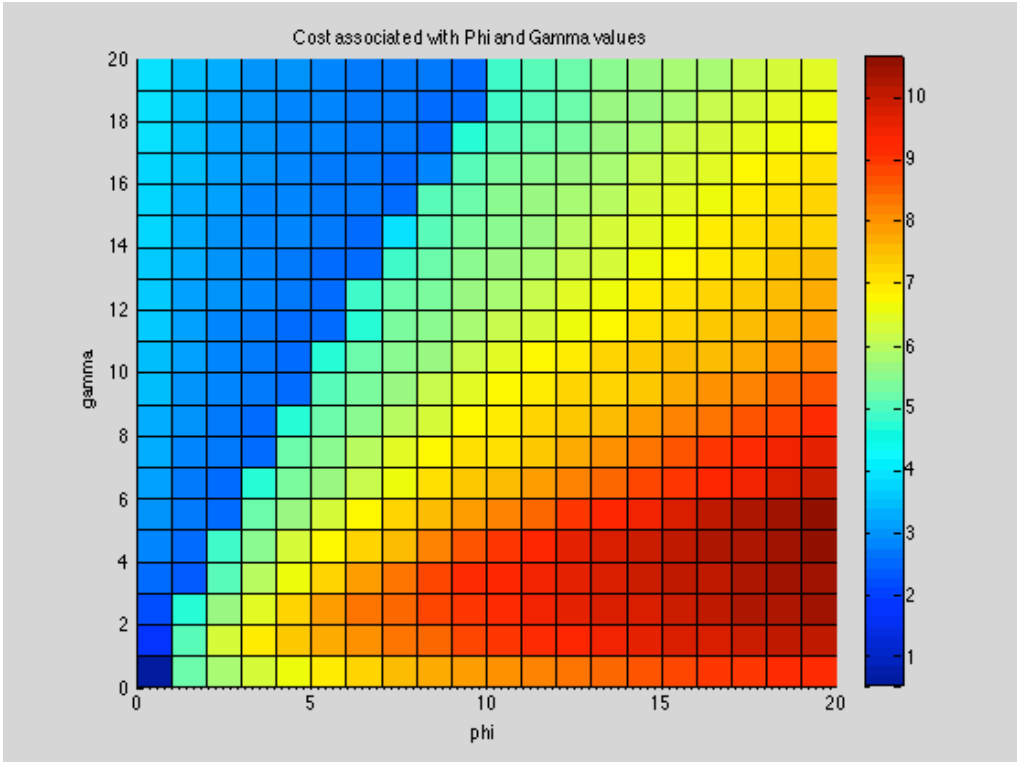

Figure 3.3. 5 Cost Analysis System 3-I

Figure 3.3.5 shows that the lowest cost region seen in blue is the same region that was seen in the previous Figure 3.3.4. Choosing values of ϕ and γ from the coolest color region in Figure 3.3.5 will result in a minimum control input and output response. It is easily noticeable in Figure 3.3.5 that the region of ϕ and γ values to yield a stable closed-loop system and disturbance accommodation also results in the lowest cost. Values of ϕ and γ were chosen for the simulation of the trajectories that are both within the lowest cost region from Figure 3.3.5 and within the region of stability and disturbance accommodation in Figure 3.3.4. Below, the output response, control input response, and the state trajectories are shown for the selected points.

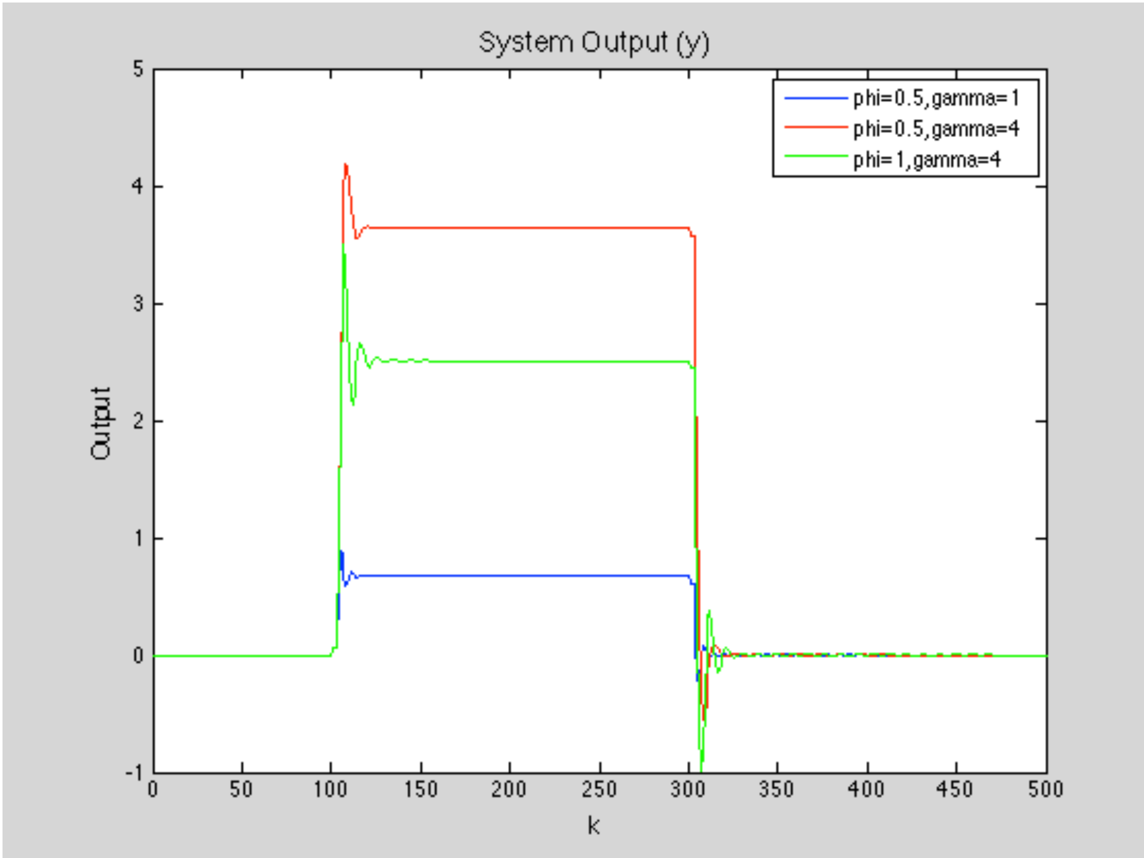

Figure 3.3. 6 System 3-I Output

As γ is increased, the magnitude of the output response increases along with a decreased maximum overshoot. As ϕ is increased, the magnitude of the output still increased, although there is a larger magnitude of the maximum overshoot as well as a longer settling time.

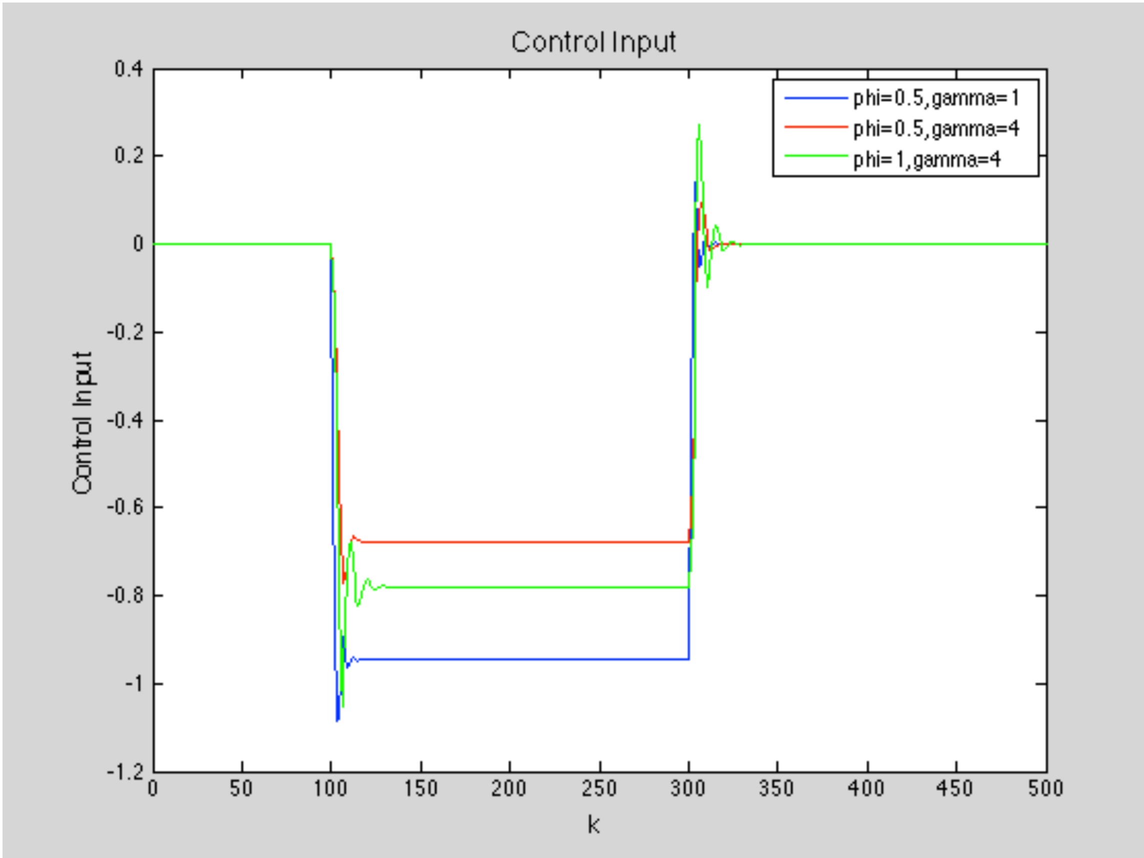

Figure 3.3. 7 System 3-I Control Input

It is visible in the control input response that as ϕ and γ are increased the magnitude of the input response decreases. Although as ϕ is increased, there is also a noticeably larger maximum overshoot and longer settling time.

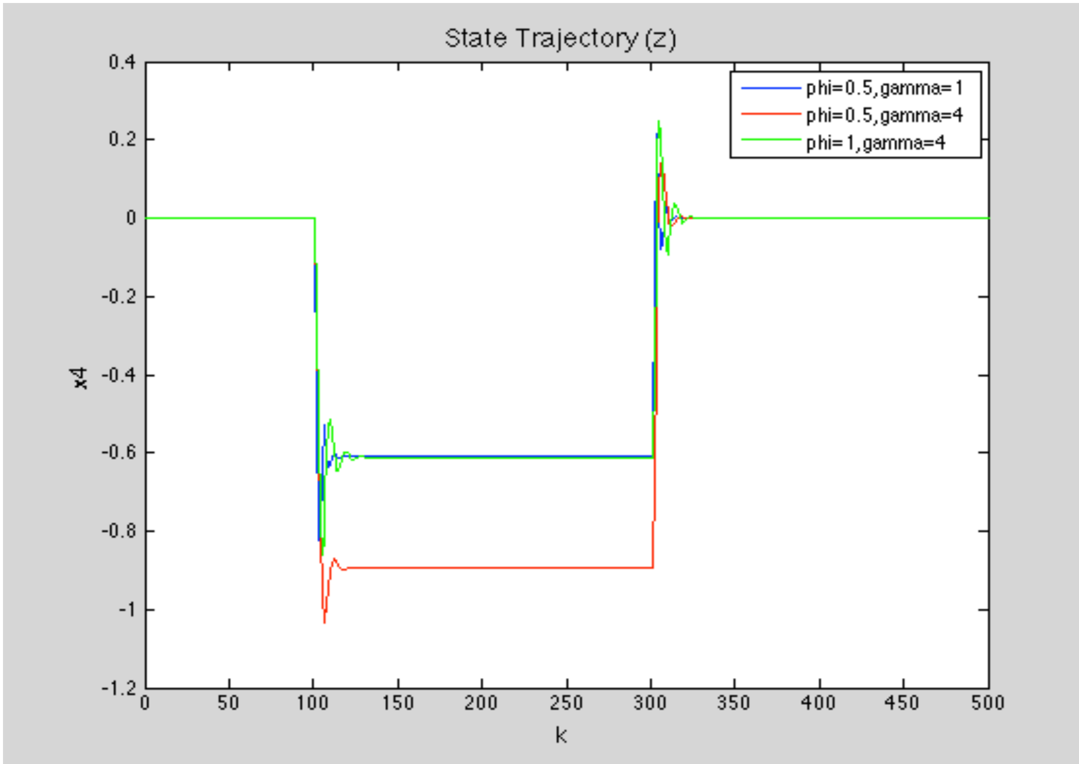

Figure 3.3. 8 System 3-I State Trajectories

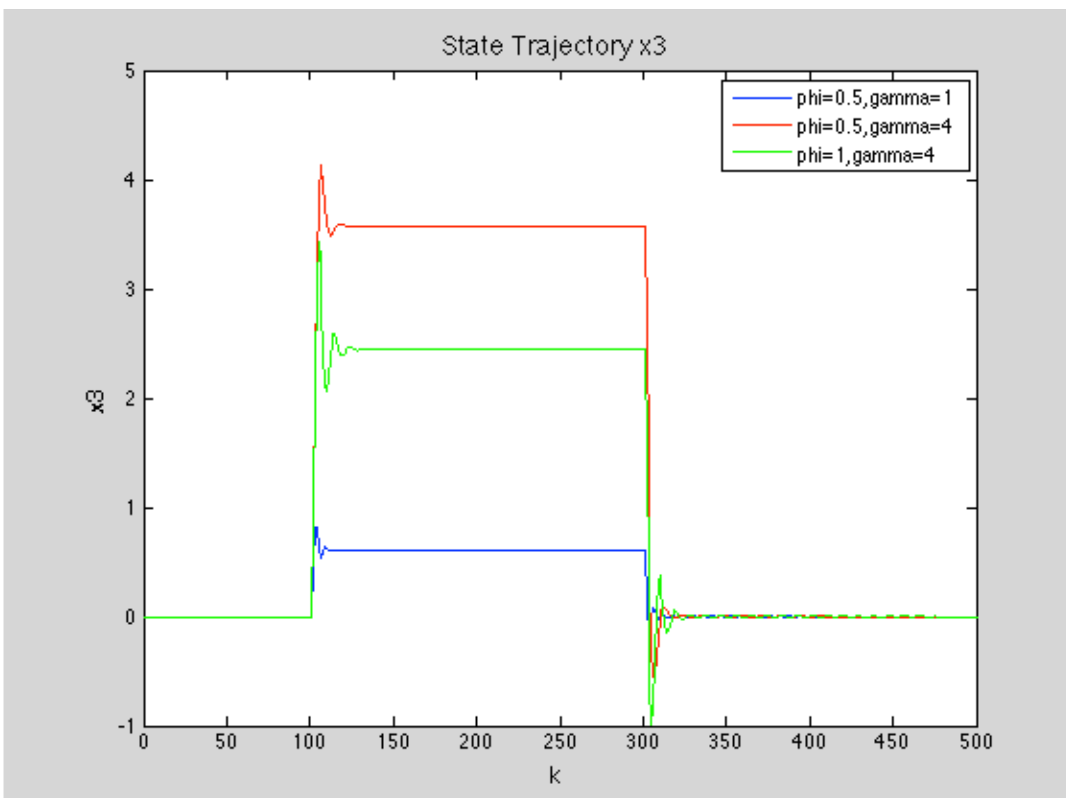

Figure 3.3. 9 System 3-I State Trajectories

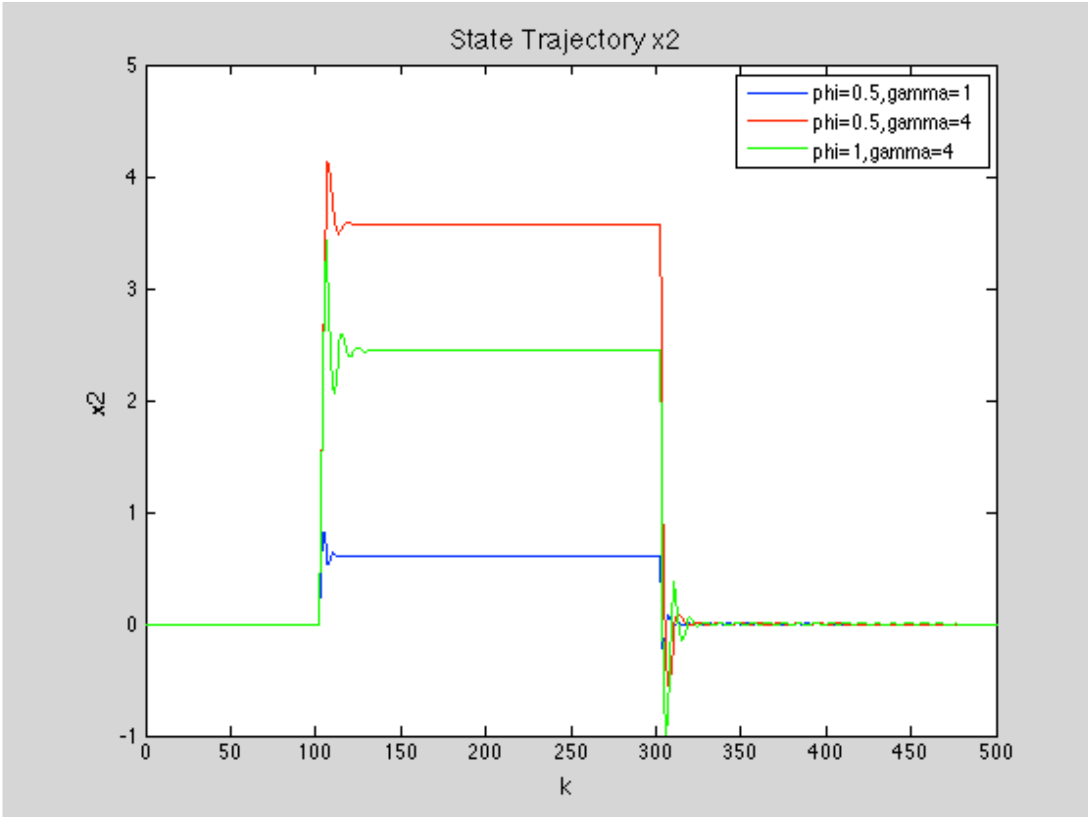

Figure 3.3. 10 System 3-1 State Trajectories

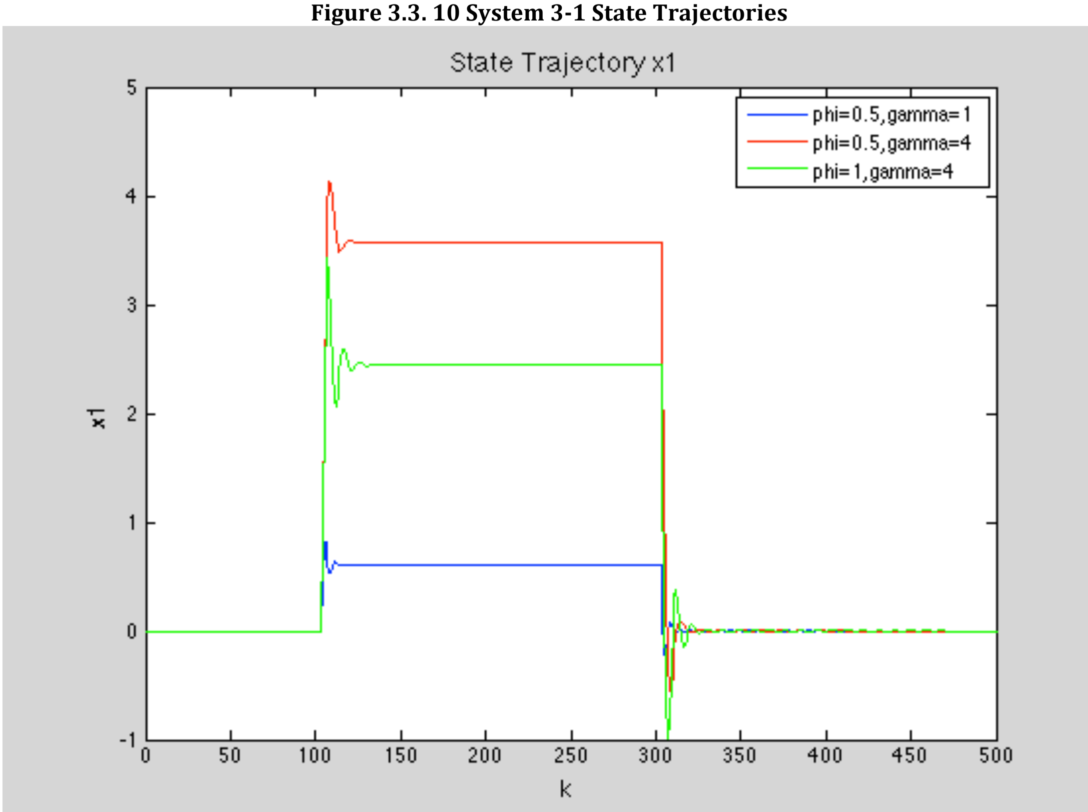

Figure 3.3. 11 System 3-1 Stat Trajectories

The state trajectories result in similar trends as the previous systems, with the magnitude of the response varying depending on the increase of either ϕ or γ. The responses in figures 3.3.2-3.3.11 show once again that the magnitude of the response and the cost increase as the open loop stability of the system decreases. Also, it can be seen that the percent overshoot and settling time of the responses decrease as the selection of the value of phi and gamma moves closer to the edge of the allowable region for closed-loop stability and disturbance accommodation.

System 3-II Evaluation

The results of the analysis of System 3-II are presented below. It should be taken into consideration that one of the eigenvalues of the open-loop system is slightly outside the unit circle, making it unstable. Knowing this, it should be expected that the allowable region of ϕ and γ to achieve closed-loop stability and disturbance accommodation will be smaller, with a higher cost required to achieve control input and output minimization.

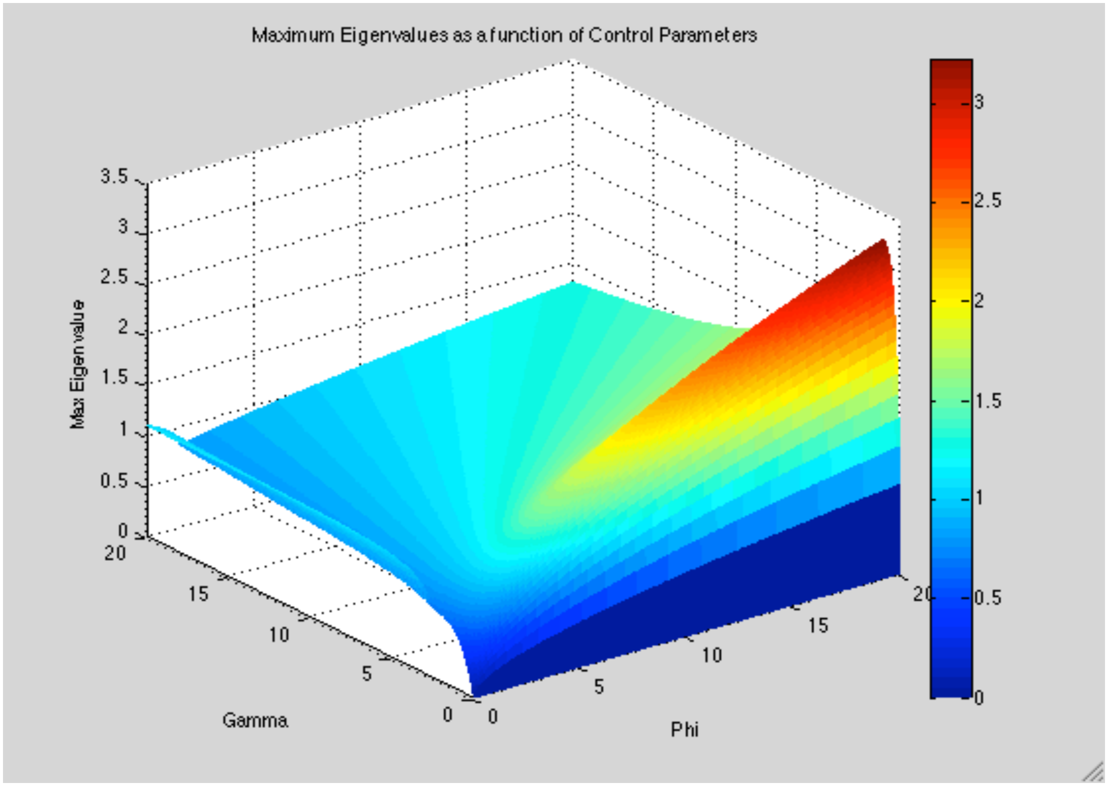

Figure 3.3. 12 System 3-II Max Closed-loop Eigenvalues versus ϕ and γ

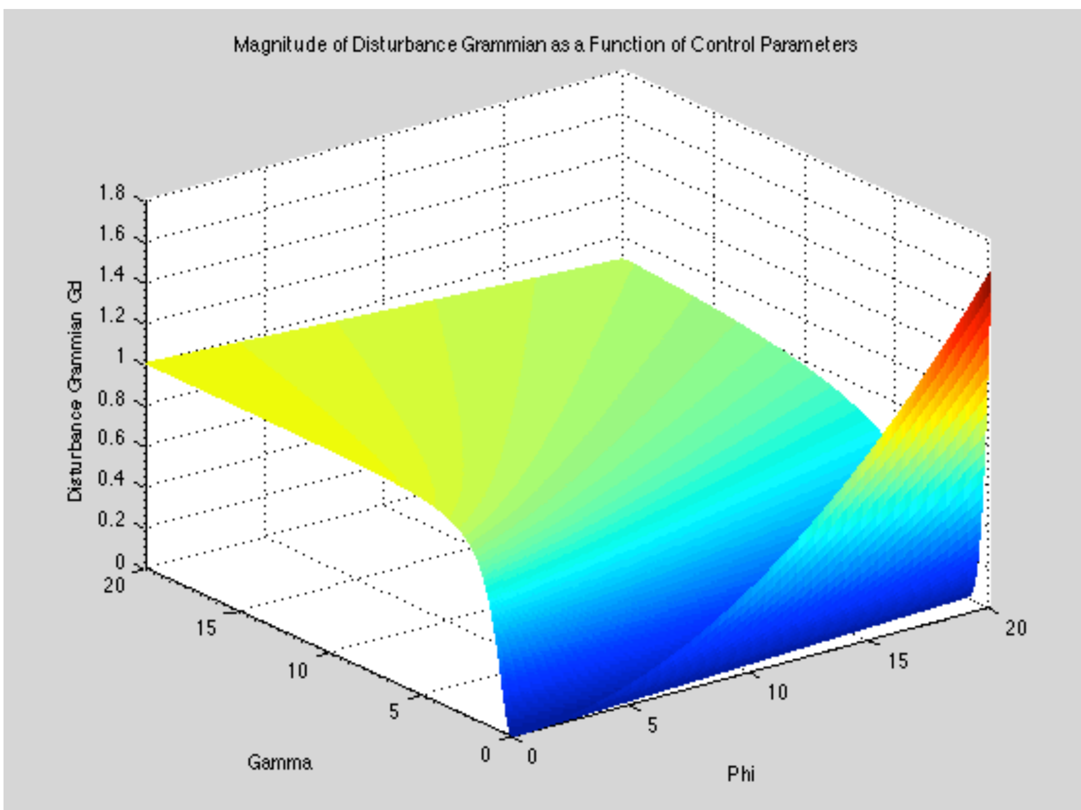

Figure 3.3. 13 System 3-II Disturbance Grammian versus ϕ and γ

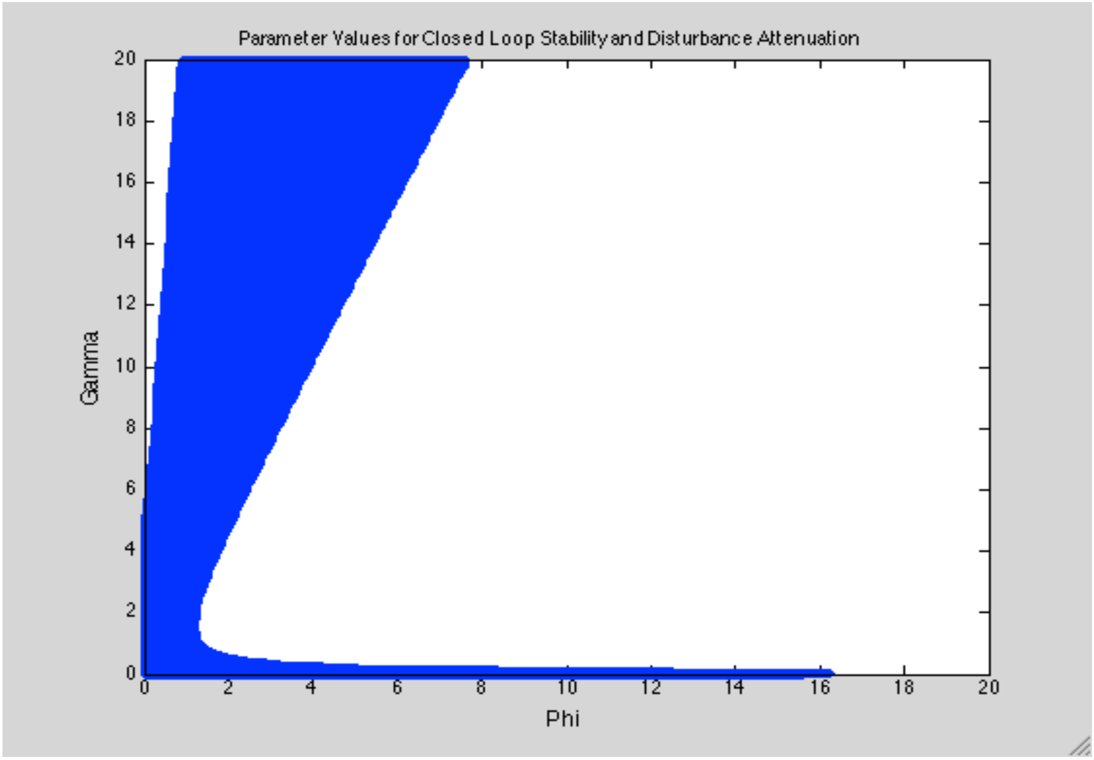

Figure 3.3. 14 Region of Closed-loop Stability and Disturbance Attenuation for System 3-II

Comparing this result to that of System 3-I, the shape of the region of acceptable values for System 3-II is noticeably similar. This region shows the values of ϕ and γ to result in a closed-loop stable system and disturbance accommodation. However, the region in Figure 3.3.14 is smaller than that of System 3-I in Figure 3.3.4. The next figure describes the cost analysis for System 3-II.

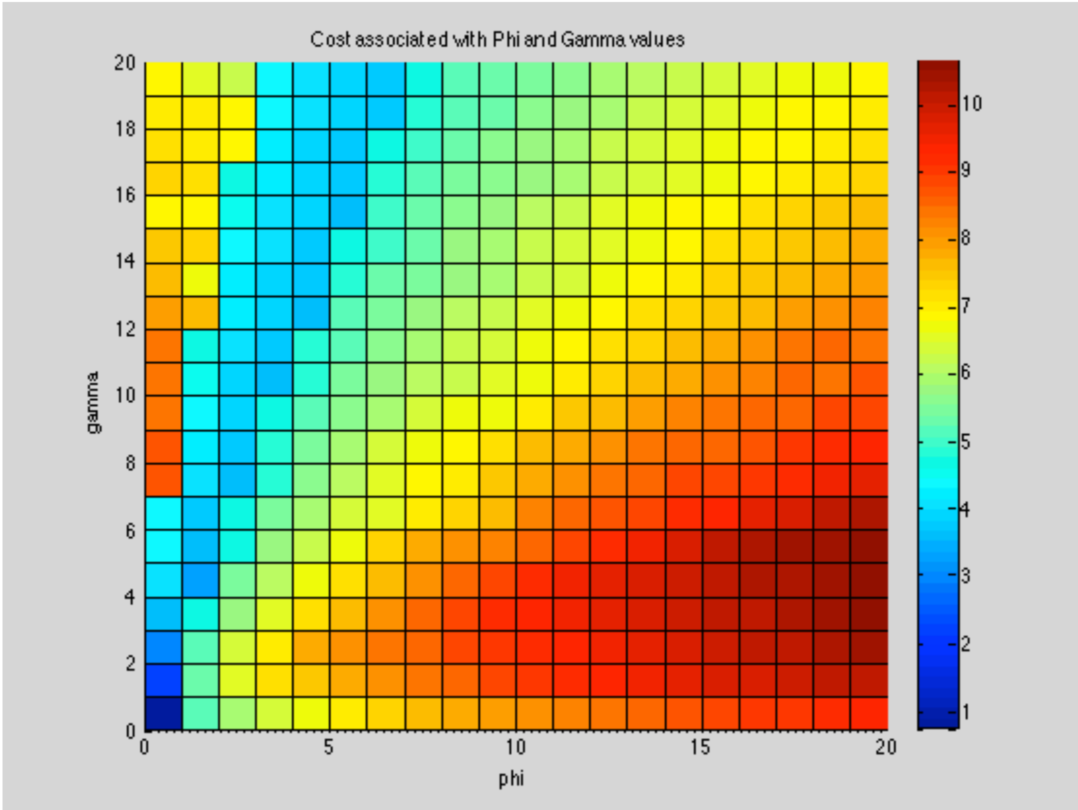

Figure 3.3. 15 Cost Analysis for System 3-II

Figure 3.3.15 shows the cost analysis of System 3-II. Taking a look at Figure 3.3.14, the region of values achieving closed-loop stability and disturbance attenuation is very similar to the region of lowest cost in Figure 3.3.15. The subsequent figures show the output *y*, input *u*, and state trajectory responses.

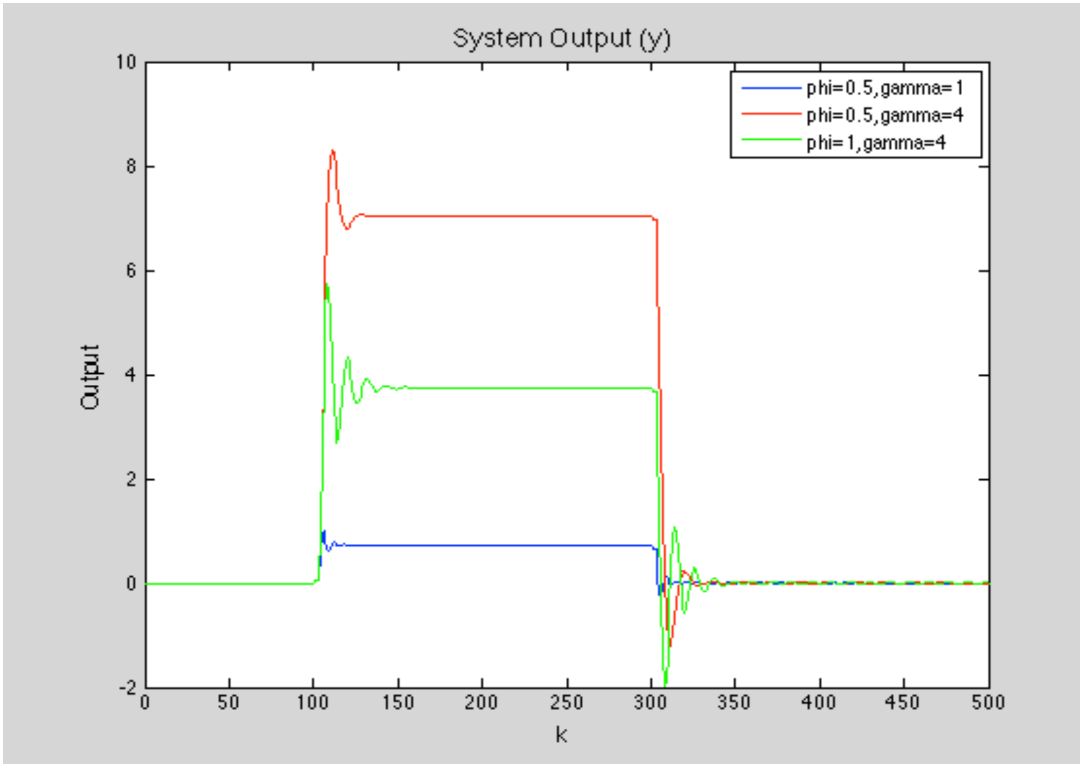

Figure 3.3. 16 System 3-II Output

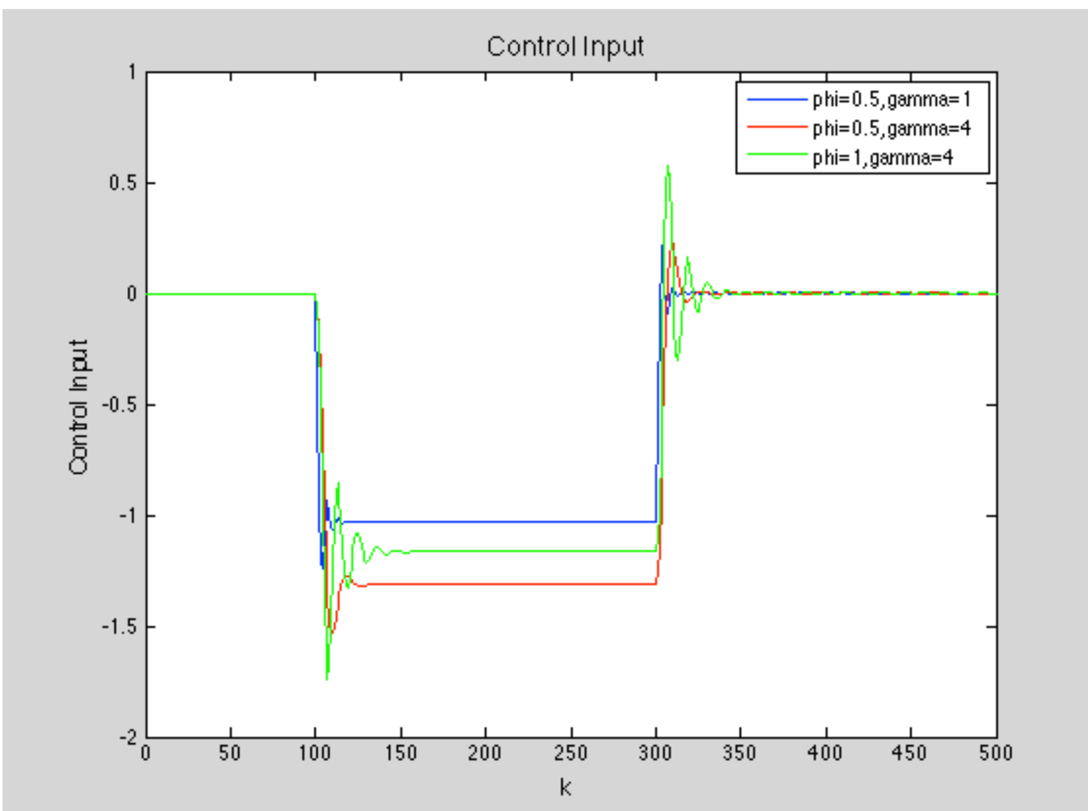

Figure 3.3. 17 System 3-II Control Input

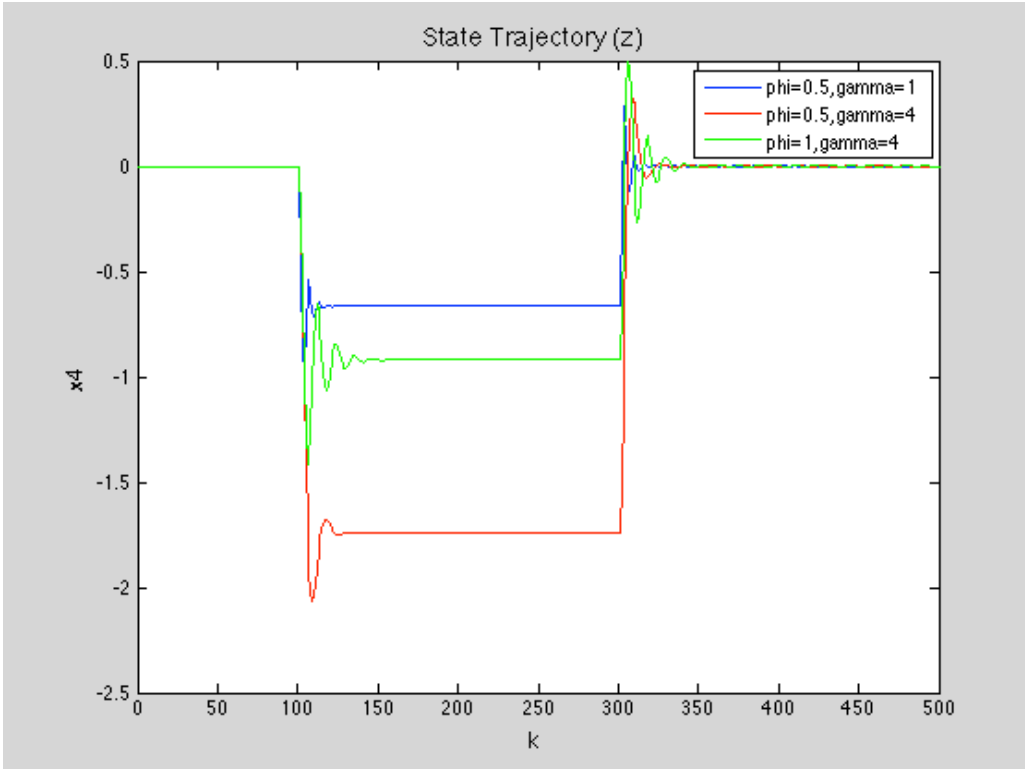

Figure 3.3. 18 System 3-II State Trajectories

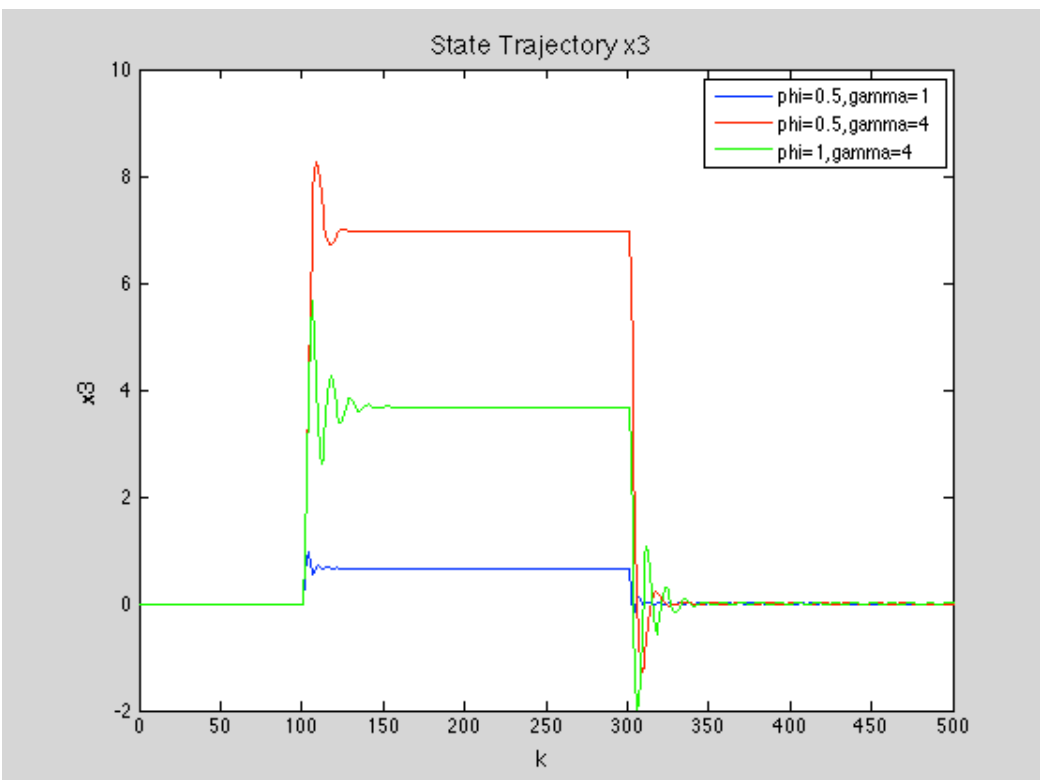

Figure 3.3. 19 System 3-II State Trajectories

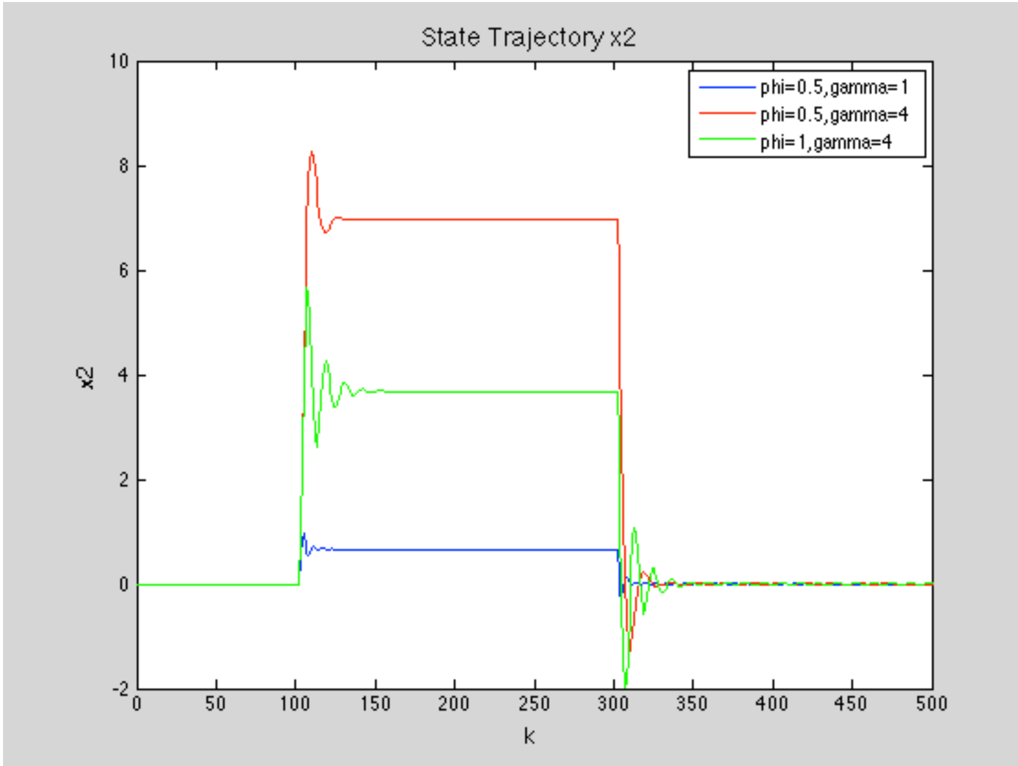

Figure 3.3. 20 System 3-II State Trajectories

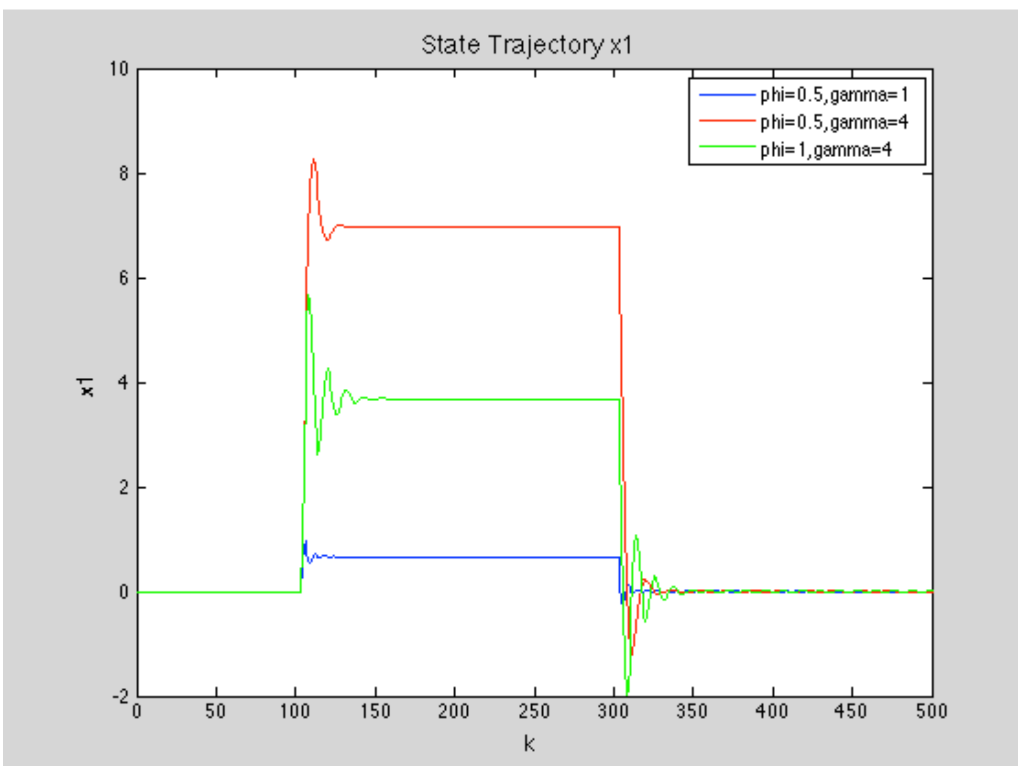

Figure 3.3. 21 System 3-II State Trajectories

It is visible that as values of ϕ and γ with a higher cost associated to them are used, a higher magnitude of the response results. The selection of ϕ also has an impact on the magnitude of the transient response and its settling time, something that could impact the implementation of this controller.

System 3-III Evaluation

The final system to be considered for the third-order case studies and this work is System 3-III, the system with its third eigenvalue farthest outside the unit circle. As a smaller acceptable region of ϕ and γ to achieve the control objectives was observed in the analysis of System II, similar results are expected from System 3-III. As this open loop system is increasingly unstable, it should require more energy and a higher control input to achieve the objectives of closed-loop stability, disturbance accommodation and control input minimization. The closed-loop eigenvalue stability analysis, disturbance Grammian versus pseudo-output weighting parameter analysis, cost function analysis and finally the responses and trajectories of the system are presented in the following figures.

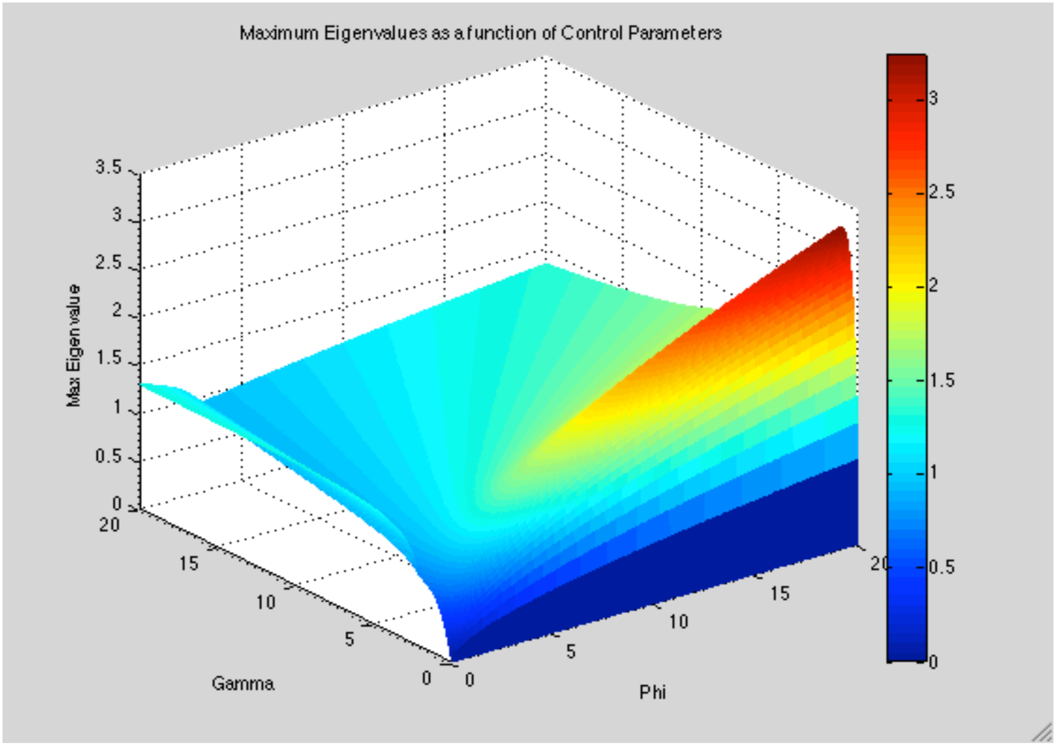

Figure 3.3. 22 System 3-III Max Closed-loop Eigenvalues versus ϕ and γ

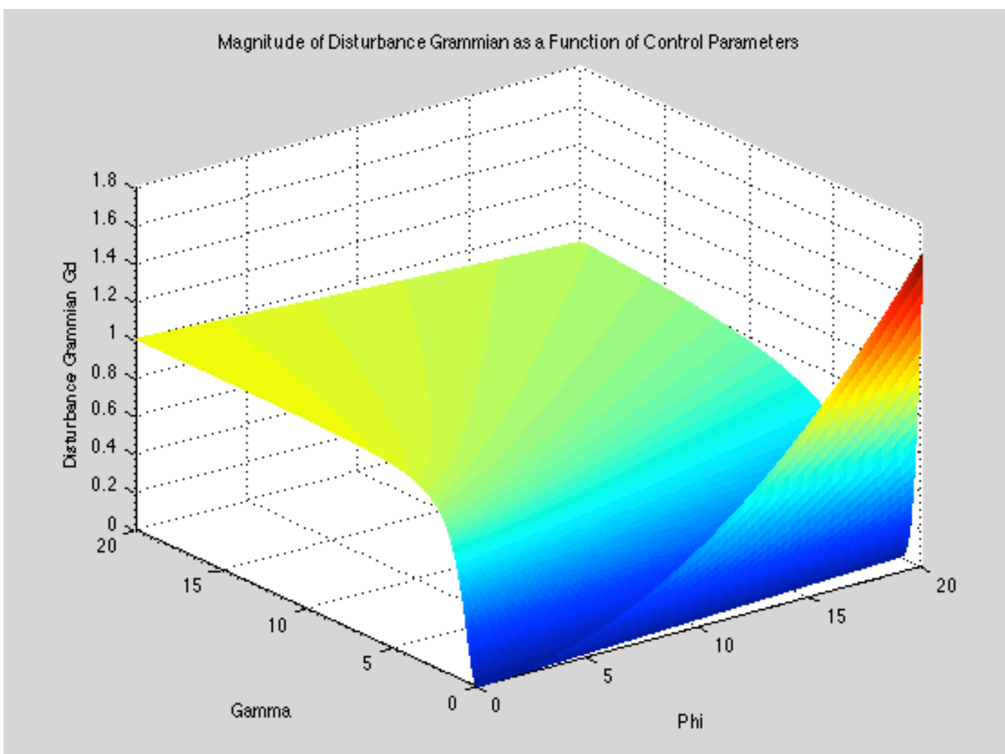

Figure 3.3. 23 System 3-III Disturbance Grammian versus ϕ and γ

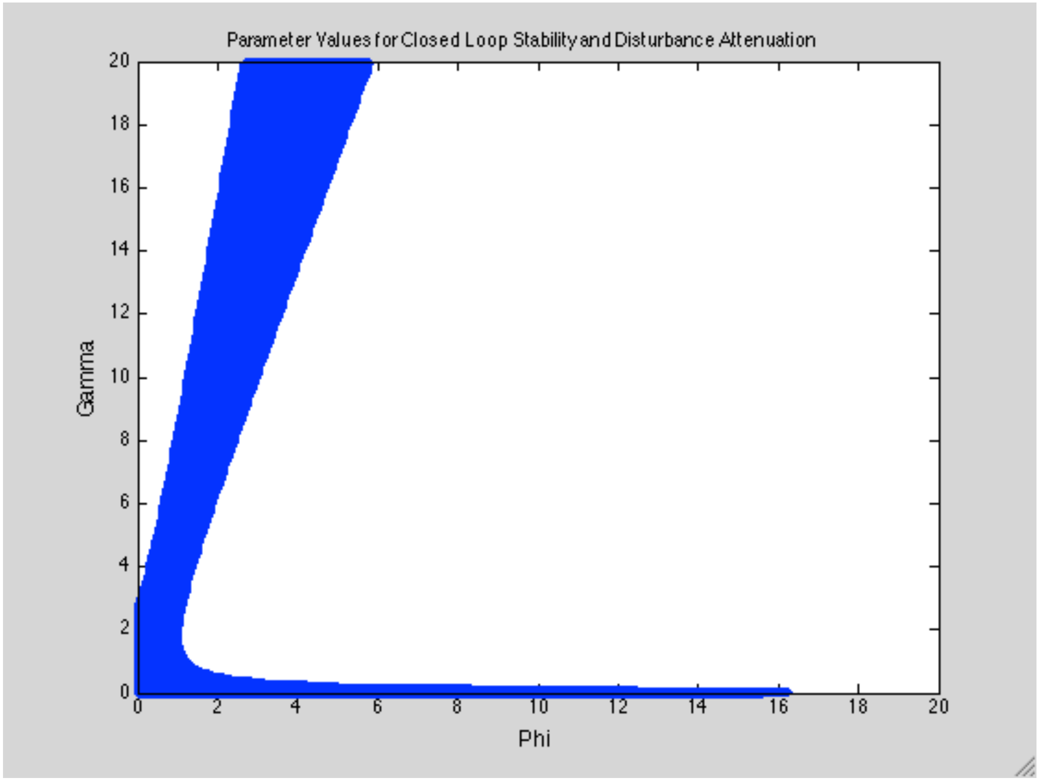

Figure 3.3. 24 Region of Closed-loop Stability and Disturbance Attenuation for System 3-III

Comparing this region of feasible ϕ and γ values to the previous regions for the third-order System 3-I and 3-II, one will notice that the region of ϕ and γ has indeed gotten smaller. This indicates that although stability and disturbance accommodation can still be achieved, there is less flexibility with the range of values that can be selected to do so.

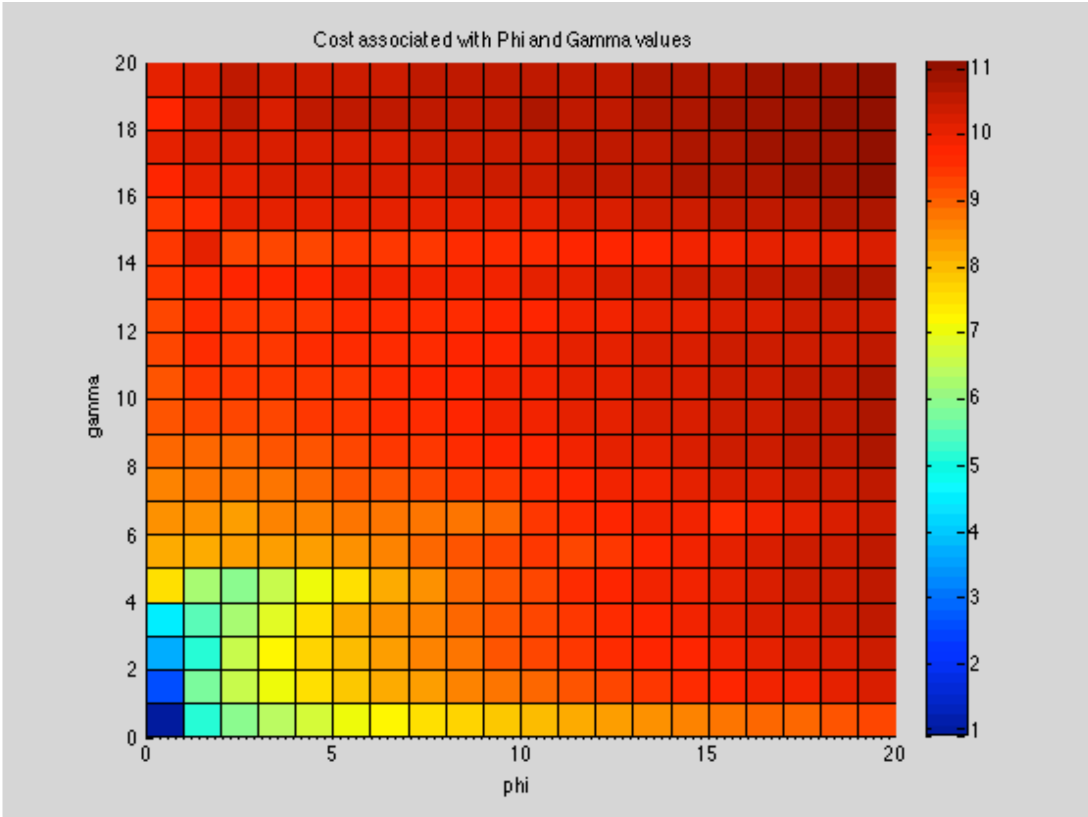

Figure 3.3. 25 Cost Analysis for System 3-III

This observation is also clearly visible in the cost function analysis, as now there is a very small region of lowest-cost associated for the simulated values of ϕ and γ. This result indicates that although an unstable open-loop system can be controlled by the use of this controller, the feasibility of implementing the controller on such systems reduces as the system increases in instability.

Continuing the analysis, the output, control input, and state trajectories are presented in the following figures.

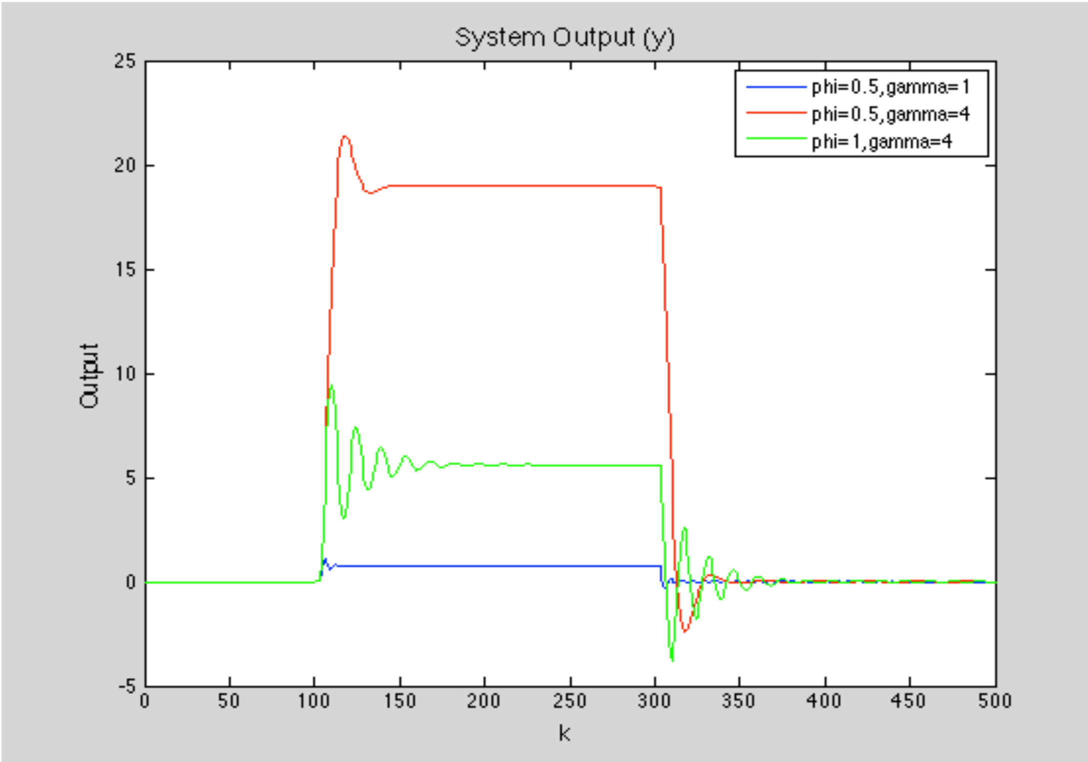

Figure 3.3. 26 System 3-III Output

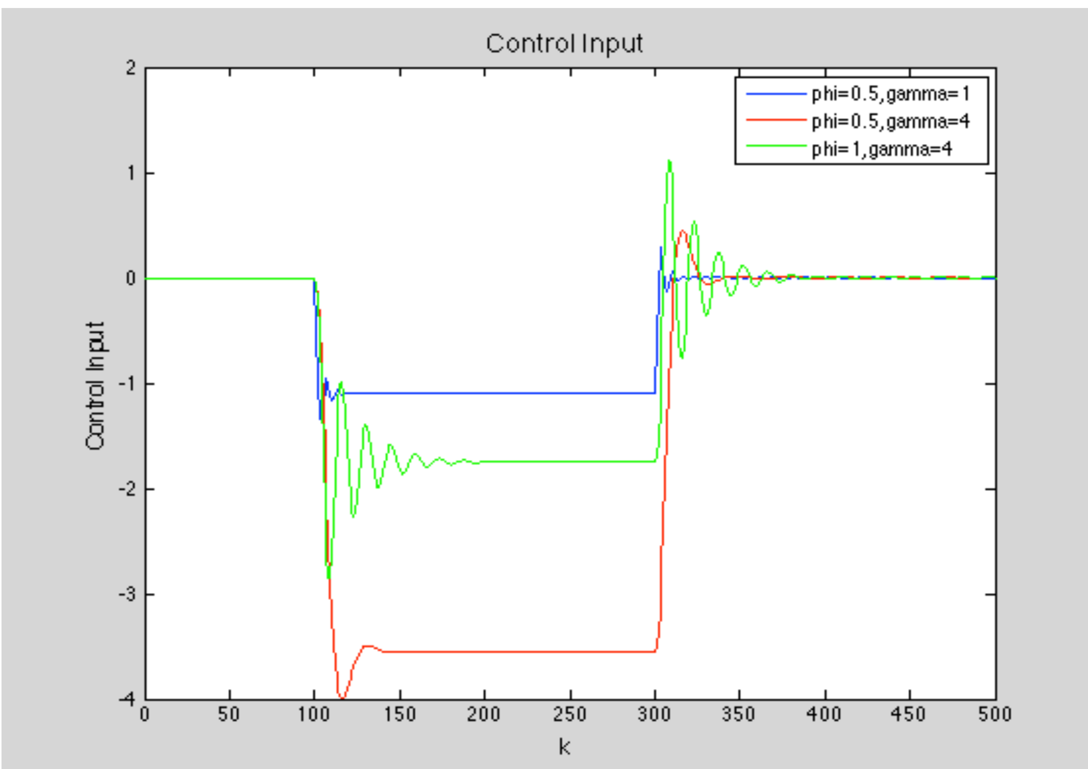

Figure 3.3. 27 System 3-III Control Input

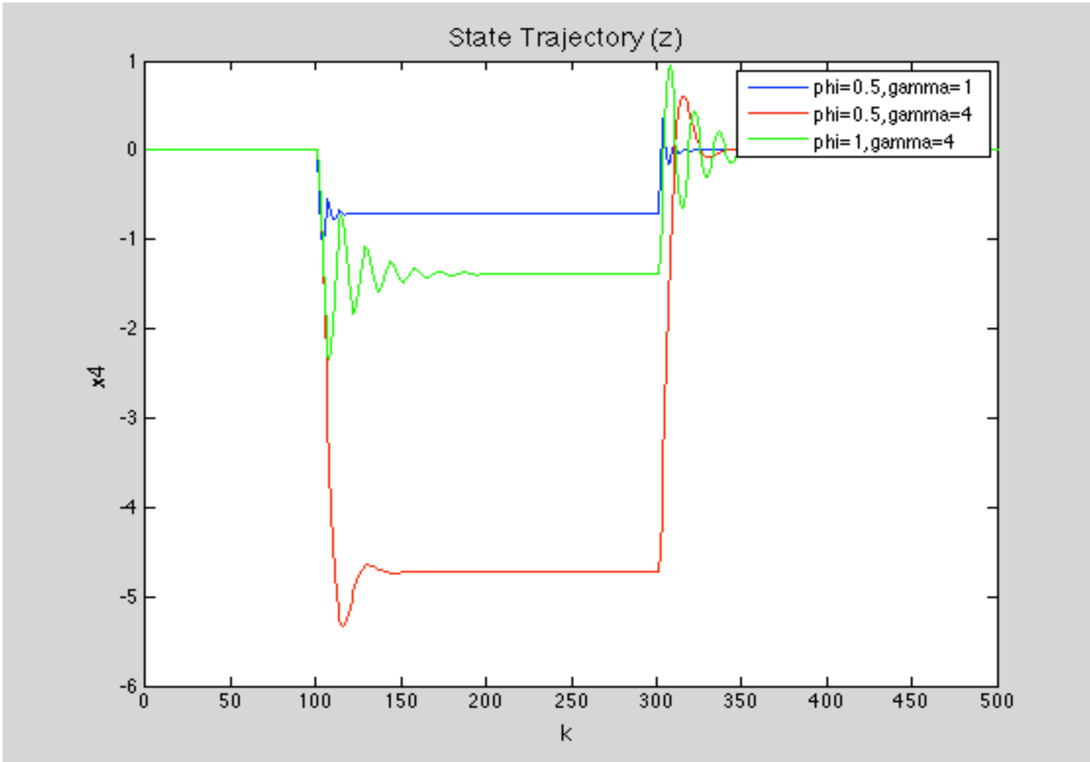

Figure 3.3. 28 System 3-III State Trajectories

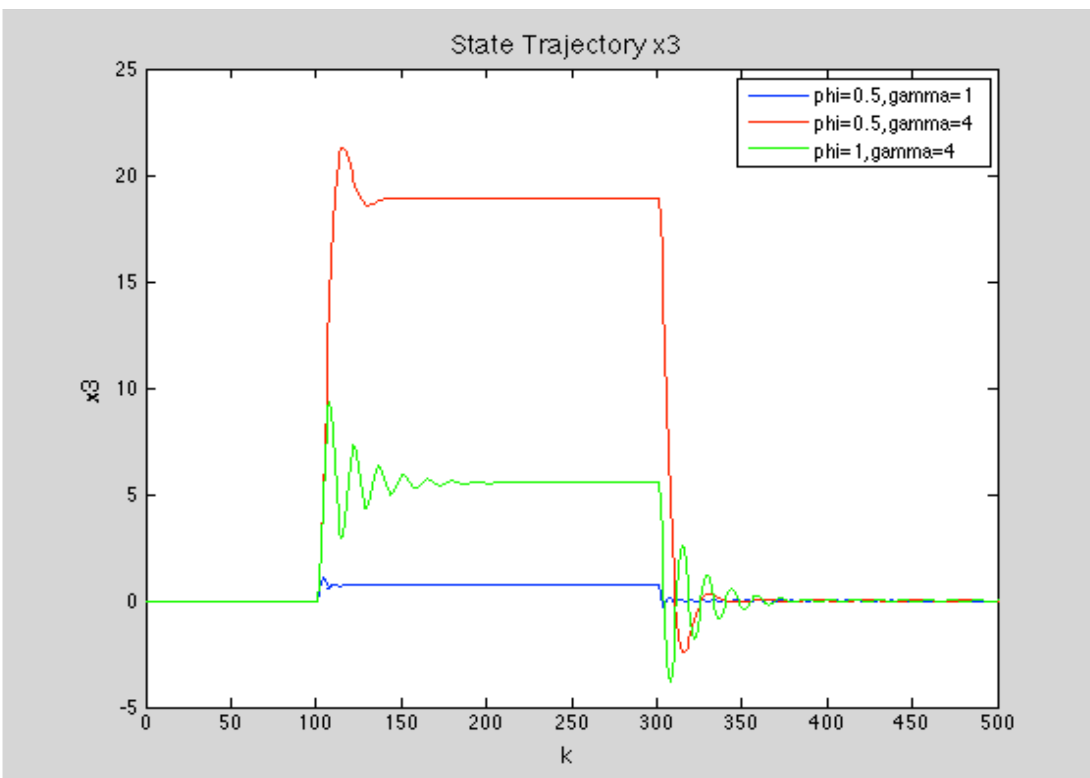

Figure 3.3. 29 System 3-III State Trajectories

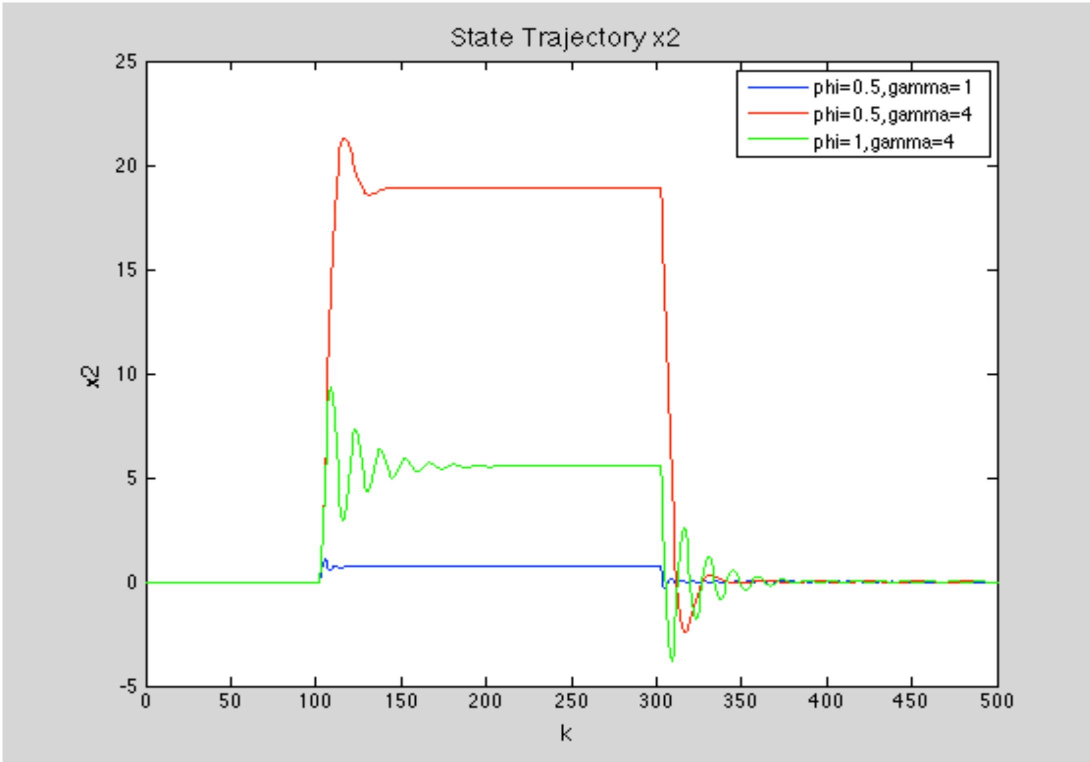

Figure 3.3. 30 System 3-III State Trajectories

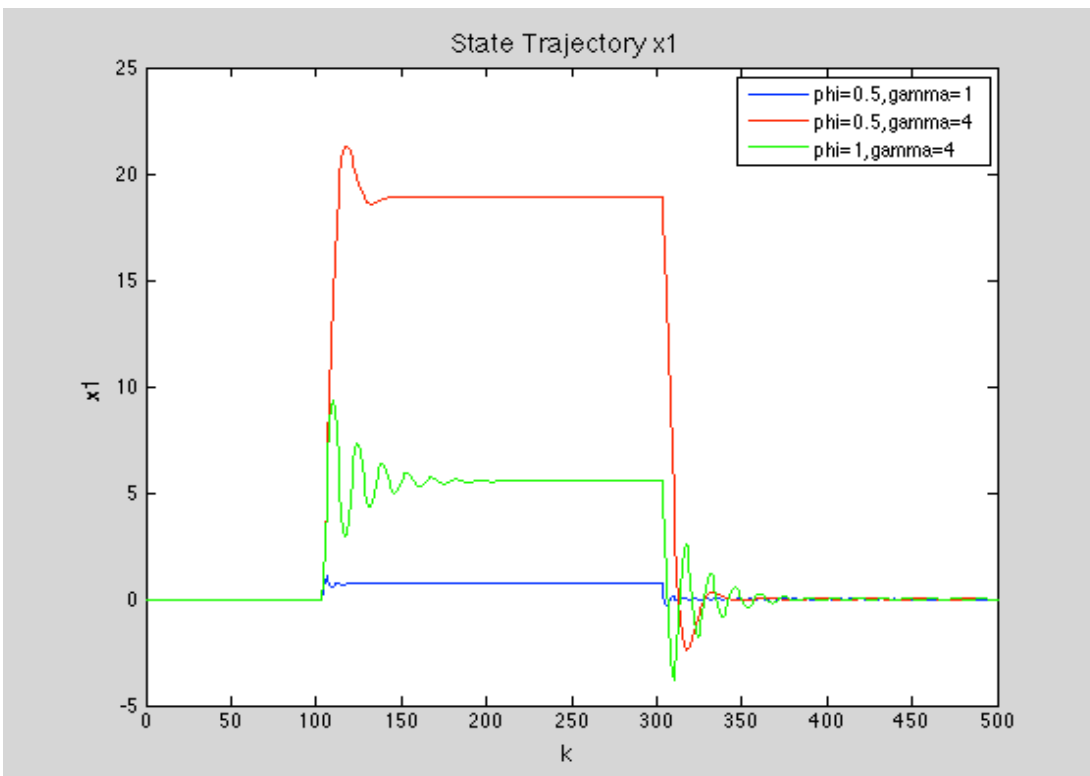

Figure 3.3. 31 System 3-III State Trajectories

Even though System 3-III has its third eigenvalues farthest outside the unit circle out of all three systems, its response still follows the observations and trends seen in the simulation of the previous systems. It can be seen that the magnitude of the response and the cost increases as the open loop stability of the system decreases. The transient response of the system responses are also affected by the selection of ϕ and γ. The percent overshoot and settling time of the responses increase as the value of ϕ and γ moves closer to the edge of the region of values to guarantee closed-loop stability and disturbance accommodation.

3.4: Overview of Results

To conclude the case studies, an overview of the results observed will be discussed. In general, it was observed that the closed-loop stability, disturbance accommodation and cost function analysis depend on the stability of the open-loop system as well as the selection of the pseudo-output weighting parameters, ϕ and γ. The acceptable range of ϕ and γ values to achieve closed-loop stability and disturbance accommodation gets smaller as each open-loop system decreases in stability. This was expected because as the open-loop system decreases in stability, it becomes increasingly difficult for the controller to achieve its objectives. The figures below show the region of acceptable ϕ and γ values to achieve closed-loop stability and disturbance accommodation as the system becomes increasingly unstable for each case study.

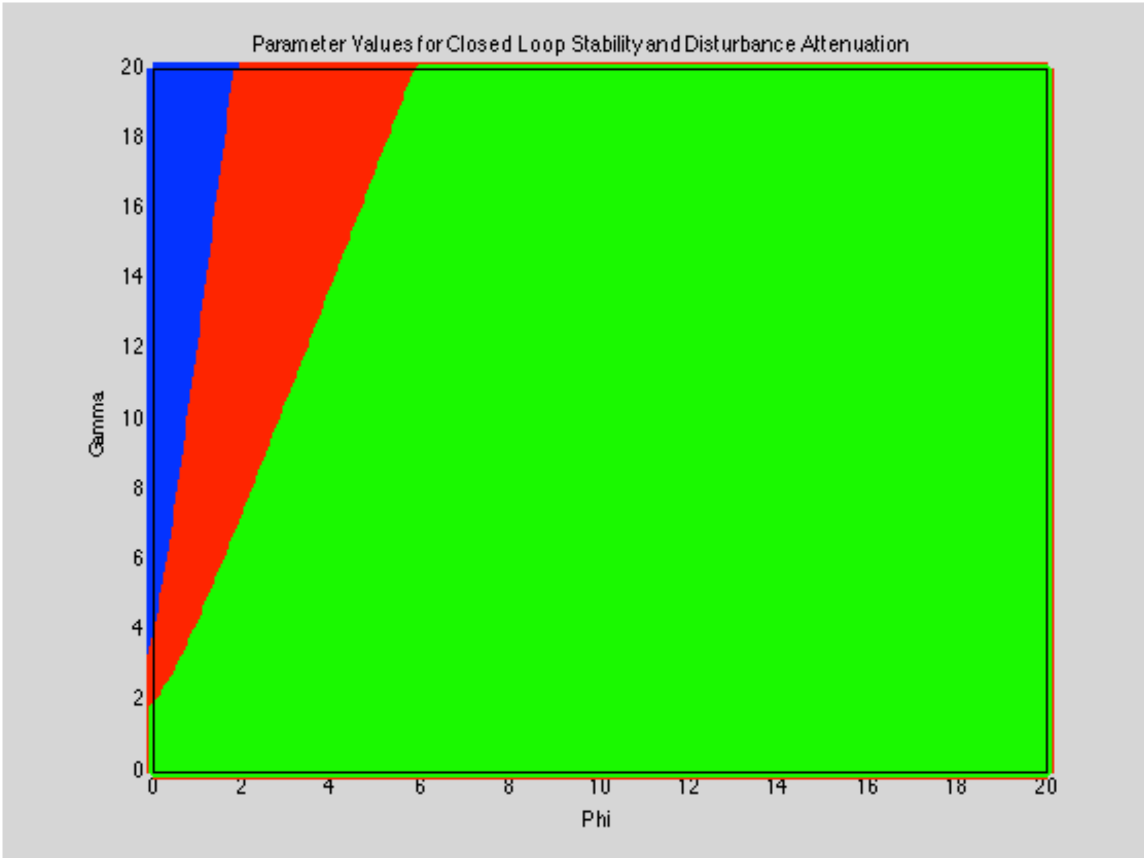

Figure 3.4. 1 First Order Systems Region of Closed loop Stability and Disturbance Attenuation

Figure 3.4.1 shows the acceptable region of ϕ and γ for the first order Systems 1-I, 1-II and 1-III. As seen in the first case study analysis, the regions for each System 1-I, System 1-II and System 1-III indeed got smaller, as is shown in blue, red, and green in the figure.

Each system in the second-order case study resulted in a smaller region of ϕ and γ values to achieve stability and disturbance accommodation. This is shown in Figure 3.4.2, where the blue region represents the stable System 2-I, the red region represents the slightly unstable System 2-II and finally the green region represents the most unstable, System 2-III.

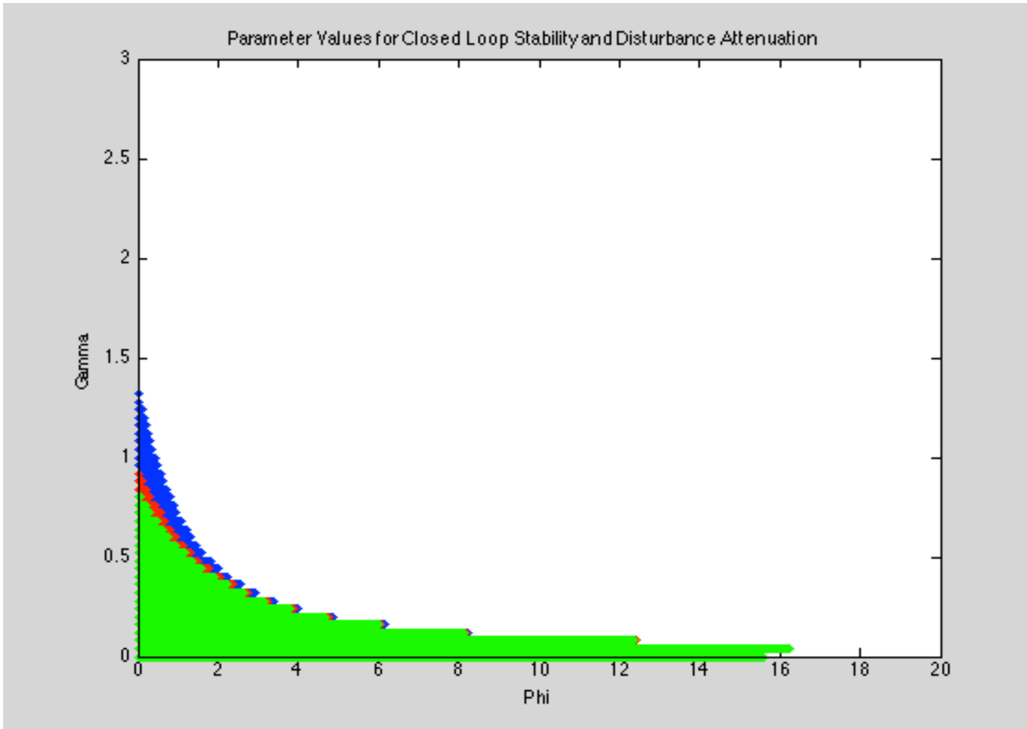

Figure 3.4. 2 Second Order Systems Region of Closed-loop Stability and Disturbance Attenuation

Lastly, the third order case study analysis resulted in the following regions of ϕ and γ values to achieve stability and disturbance accommodation. Once again, the blue system represents System 3-I, the red region represents System 3-II and the green region represents System 3-III.

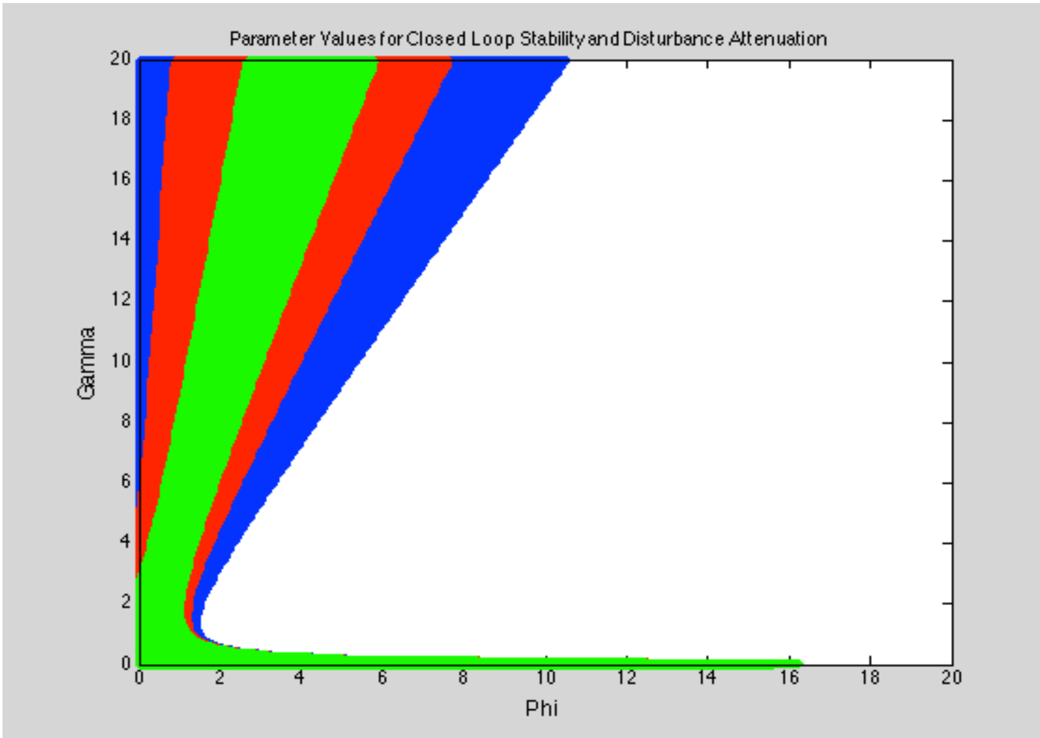

Figure 3.4. 3 Third Order Systems Region of Closed-loop Stability and Disturbance Attenuation

It was also observed that the regions of ϕ and γ associated with the lowest cost for implementing particular ϕ and γ values in the cost function analysis decreased as the stability of each open-loop system decreased. The results in the analysis of the third-order systems show that the values of ϕ and γ that result in both closed-loop stability and disturbance accommodation also happen to be the same values that result in a lowest cost from the cost analysis.

Not only were trends in the stability, disturbance accommodation, and cost analysis observed, but also seen in the selection of ϕ and γ values for the simulated input, output and state trajectory responses. There were also trends noted in the maximum overshoot value of the system. In general, the magnitude of the system response and associated cost increased as the open-loop stability of the system decreased. Furthermore, the percent overshoot and settling time of the transient response of the systems increased as the selection of the ϕ and γ values moved closer to the edge of the acceptable region of values.

Tables 7-9 below summarize the results seen in the analyses of the first, second and third order systems. The tables show the cost associated with each particular set of ϕ and γ values chosen from the region of values yielding in a stable system and disturbance accommodation, the magnitude of the steady state input and steady state output responses, as well as the maximum percent overshoot and settling time. The cost value was determined from the cost function defined in Chapter 2, except for that the base-10 logarithm was not taken, as the trends are the same.

System	ϕ	γ	Cost	$\|u_{ss}\|$	$\|y_{ss}\|$	Max P.O.	T_s	Max P.O.	T_s
						u(k)	u(k)	y(k)	y(k)
1-I	2	2	0.1346	0.712	0.64	--	6	--	6
	4	8	1.8819	0.521	1.022	--	8	--	9
	4	2	0.6918	0.8312	0.4015	--	3	--	5
1-II	2	2	5.801	1.091	0.9772	--	11	--	9
	4	8	100.23	1.236	2.423	--	25	--	25
	4	2	1.9738	1.044	0.5037	--	9	--	6
1-III	2	2	14.9073	1.34	1.199	--	16	--	14
	4	8	2047.8588	1.152	0.5707	--	11	--	11
	4	2	3.0395	1.147	0.553	--	9	--	10

Table 7: First-order system summary of results

Table 7 shows the summary of results for the first order systems 1-I, 1-II, and 1-III. For these particular systems, the responses shown previously indicate that there is no significant percent overshoot and settling time transient response associated with it. As the particular values of ϕ and γ that were selected for these first order systems are relatively close to the origin and not the edge of the allowable region, the output shows almost no overshoot. Considering the cost values associated with each particular system, it can be seen that the cost increases as each open-loop system becomes less stable. This is expected, as it will require more energy to stabilize a system that is not originally stable in the open-loop case. Furthermore, the magnitude of the input and output responses also increases as each open loop system becomes less stable. This implies that although the controller will succeed in reducing the effect of the disturbance on the system, its effectiveness will depend on the open-loop stability of the system as well as the selection of the pseudo-output weighting parameters ϕ and γ.

Table 8 shows the summary of results for the second-order systems 2-I, 2-II, and 2-III.

System	ϕ	γ	Cost	$\lvert u_{ss} \rvert$	$\lvert y_{ss} \rvert$	Max P.O.	T_s	Max P.O.	T_s
						u(k)	u(k)	y(k)	y(k)
2-I	1	0.2	0.8939	0.9605	0.08535	1.40%	5	5.59%	5
	1	0.4	0.6098	0.8534	0.1432	3.60%	14	25.54%	12
	4	0.2	1.0804	1.011	0.05802	0.40%	17	10.38%	27
2-II	1	0.2	0.8939	0.9565	0.08522	1.88%	5	5.81%	5
	1	0.4	0.6317	0.8406	0.1418	9.04%	14	26.95%	15
	4	0.2	1.1008	1.012	0.05805	0.69%	17	10.19%	34
2-III	1	0.2	0.8939	0.9545	0.08515	2.04%	5	5.79%	4
	1	0.4	0.648	0.8343	0.1411	10.85%	15	27.50%	24
	4	0.2	1.111	1.013	0.05806	0.79%	23	10.17%	37

Table 8: Second-order system summary of results

In the responses for systems 2-I, 2-II, and 2-III it can be seen that there is a visible transient response associated with particular values of ϕ and γ. Figure 3.4.2 shows that the region of acceptable ϕ and γ values to guarantee closed-loop stability and disturbance accommodation is much smaller than for the other 1st and 3rd order systems, therefore the selection of ϕ and γ values and their proximity to the edge of the region will result in a greater transient response. In general, the second order systems show that the cost associated with the ϕ and γ values increase as the open loop systems decrease in stability.

Lastly, Table 9 shows the summary of results for each of the third order systems 3-I, 3-II, and 3-III.

System	**ϕ**	**γ**	**Cost**	**$\lvert u_{ss} \rvert$**	**$\lvert y_{ss} \rvert$**	**Max P.O.**	**T_s**	**Max P.O.**	**T_s**
						u(k)	**u(k)**	**y(k)**	**y(k)**
3-I	0.5	1	3.49	0.9452	0.6728	14.79%	19	31.81%	22
	0.5	4	791.04	0.6783	3.639	13.65%	28	15.20%	28
	1	4	381.86	0.78	2.509	34.87%	40	39.50%	38
3-II	0.5	1	5.58	1.03	0.7293	19.03%	24	39.31%	23
	0.5	4	18496.94	1.313	7.025	16.76%	40	18.39%	38
	1	4	3717.92	1.166	3.743	49.66%	61	52.91%	85
3-III	0.5	1	8.02	1.096	0.7732	26.09%	25	47.31%	17
	0.5	4	1439129.91	3.551	18.96	12.06%	54	12.59%	45
	1	4	36694.42	1.747	5.608	63.65%	133	68.71%	106

Table 9: Third-order system summary of results

It can be seen that the cost of each particular ϕ and γ value as well as the magnitude of the input and output responses increases as the stability of the open loop system decreases. Furthermore, it can be noted that the percent overshoot and settling time increases as the values of ϕ and γ get closer to the edge of the acceptable region of values to guarantee closed-loop stability and disturbance attenuation.

These results could be very beneficial for the implementation of this controller, as particular systems may have certain needs or requirements that could be ensured by the proper choice of ϕ and γ in this work. In the next chapter, other considerations for the implementation of this controller and systems will be discussed.

Chapter 4: Considerations for Implementation

When implementing the controller proposed in this work, the system provided will usually not be in discrete-time canonical form as utilized in the proposed control technique. Since discretization methods are simply different mathematical representations of continuous-time systems, the results of implementing this control design depend on the discretization method utilized. In this chapter, the discretization method and canonical form that is used to arrive at the system model described in Chapters 2 and 3 will be discussed. In order to guarantee that the continuous time system parameters will not infringe upon the stability analysis of the system, Jury's Stability Test conditions will be considered and applied to first, second, and third-order systems.

In addition to discussing the aforementioned considerations for implementation, this chapter will also present comparisons between the controller proposed in this work for systems *without* a control term in the output and a previously developed deadbeat disturbance accommodation controller for systems *with* a control term present in the output [11]. In the deadbeat DAC design for systems with a control term in the output, the state of the system $x(k+1)$ is augmented with the output $y(k)$ to result in the composite system used in the design of the controller and disturbance gains. In the design proposed in this work, systems specifically without a control term present in the output are considered, introducing a pseudo-output with a factor of the control input into the system. The newly defined pseudo-output is augmented with the state update equation $x(k+1)$ to result in the composite system as dictated in Chapter 2. Upon comparison of the two controllers, the benefits of implementing the control method presented in this work for systems lacking a control term in the output will be discussed.

To address these considerations for implementation, a first-order, second-order, and third-order system will be presented. Although the systems will be presented in general, in actuality they could represent any variety of practical systems modeled as either first, second, or third order systems. It is crucial that the continuous-time system model also lack a control term in the output equation, as the introduction of a control term in the output when it is not originally present is one of the primary benefits of implementing this control technique.

4.1: Discrete-time Systems

When modeling a particular problem or practical system for use in a controller development, it will often be in continuous-time. For each system, Forward-Euler discretization methods will be applied and the systems put into canonical form. Once the systems are in discrete-time canonical form, the controller development can be continued as seen in Chapter 2 and Chapter 3.

4.1.1: First-order Systems

To begin, the first-order system will be considered. A generic, continuous-time model in canonical form will be used (4.1-4.2), where $\overline{A}$, $\overline{B}$, $\overline{F}$, $\overline{C}$, and $\overline{G}$ represent the corresponding coefficient matrices in canonical form, *x(t)* is the state variables of the system, *u(t)* is the control input, *y(t)* is the output, and *w(t)* is the applied disturbance signal, respectively.

$$\dot{x}(t) = \overline{A}x(t) + \overline{B}u(t) + \overline{F}w(t) \tag{4.1.1}$$

$$y(t) = \overline{C}x(t) + \overline{G}w(t) \tag{4.1.2}$$

$$\overline{A} = a,\ \overline{B} = 1,\ \overline{F} = 1,\ \overline{C} = 1,\ \overline{G} = g_o$$

As this system is of first-order, the resulting coefficients are scalar values. In order to design and apply the controller presented in this work, the system must be discretized. Forward-Euler methods have been used to discretize the system. The first-order continuous time system is discretized below, where T is the sampling time and I_1 is the first order identity matrix.

$$x(k+1) = (I_1 + T\overline{A})x(k) + T\overline{B}u(k) + T\overline{F}w(k) \tag{4.1.3}$$

$$y(k) = Cx(k) + Gw(k) \tag{4.1.4}$$

Now that the original continuous-time system has been put into the discrete-time canonical form, the control technique described in Chapter 2 and illustrated in Chapter 3 can be implemented.

4.1.2: Second-order Systems

The second-order system model is presented, where $\overline{A}, \overline{B}, \overline{F}, \overline{C}$, and $\overline{G}$ are represented in canonical form and $\overline{G}$ is a scalar value. The A matrix parameters a_1 and a_2 represent parameters from the original modeled system.

$$\dot{x}(t) = \overline{A}x(t) + \overline{B}u(t) + \overline{F}w(t) \tag{4.1.5}$$

$$y(t) = \overline{C}x(t) + \overline{G}w(t) \tag{4.1.6}$$

$$\overline{A} = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix},\ \overline{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \overline{F} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \overline{C} = \begin{bmatrix} 1 & 0 \end{bmatrix},\ \overline{G} = g_0$$

Once again, forward-Euler discretization methods were applied to the continuous-time system to result in the second-order discrete-time system for the sampling time T and second-order identity matrix I_2.

$$x(k+1) = A_D x(k) + B_D u(k) + F_D w(k) \tag{4.1.7}$$

$$y(k) = Cx(k) + Gw(k)$$

$$A_D = \left(I_2 + T\overline{A}\right) = \begin{bmatrix} 1 & T \\ -Ta_1 & 1 - Ta_2 \end{bmatrix} \tag{4.1.8}$$

$$B_D = T\overline{B} = \begin{bmatrix} 0 \\ T^2 \end{bmatrix} \quad , \ F_D = T\overline{F} = \begin{bmatrix} 0 \\ T \end{bmatrix}$$

Now, the canonical form of the discrete-time system can be determined simply as shown below where $\alpha_1 = 2 - Ta_2, \ \alpha_2 = -T^2 a_1 + Ta_2 - 1$.

$$x(k+1) = \overline{A}x(k) + \overline{B}u(k) + \overline{F}w(k) \tag{4.1.9}$$

$$y(k) = \overline{C}x(k) + \overline{G}w(k) \tag{4.1.10}$$

$$\overline{A} = \begin{bmatrix} 0 & 1 \\ -\alpha_1 & -\alpha_2 \end{bmatrix}, \overline{B} = \begin{bmatrix} 0 \\ T^2 \end{bmatrix}, \overline{F} = \begin{bmatrix} 0 \\ T^2 \end{bmatrix}, \overline{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \overline{G} = g_0$$

Now that the original continuous-time system is in discrete-time canonical form, the control technique in Chapters 2 and 3 can be performed.

4.1.3: Third-order Systems

Similar to the previous two systems, the third order continuous-time system model in canonical form is shown, along with its corresponding transfer function.

$$\dot{x}(t) = \overline{A}x(t) + \overline{B}u(t) + \overline{F}w(t) \tag{4.1.11}$$

$$y(t) = \overline{C}x(t) + \overline{G}w(t) \tag{4.1.12}$$

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix},\ \bar{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ \bar{F} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ \bar{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},\ \bar{G} = g_0$$

The continuous-time system is discretized once again according to the forward-Euler discretization method, for the sampling time T and identity matrix I_3

$$x(k+1) = A_D x(k) + B_D u(k) + F_D w(k) \tag{4.1.13}$$

$$y(k) = Cx(k) + Gw(k) \tag{4.1.14}$$

$$A_D = \left(I_3 + T\bar{A}\right) = \begin{bmatrix} 1 & T & 0 \\ 0 & 1 & T \\ -Ta_1 & -Ta_2 & 1-Ta_3 \end{bmatrix},\ B_D = T\bar{B} = \begin{bmatrix} 0 \\ 0 \\ T \end{bmatrix},\ F_D = T\bar{F} = \begin{bmatrix} 0 \\ 0 \\ T \end{bmatrix}$$

The discrete-time canonical form is shown below, where α_1, α_2, and α_3 are defined as follows.

$$x(k+1) = \bar{A}x(k) + \bar{B}u(k) + \bar{F}w(k) \tag{4.1.15}$$

$$y(k) = \bar{C}x(k) + \bar{G}w(k) \tag{4.1.16}$$

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_1 & -\alpha_2 & -\alpha_3 \end{bmatrix}, \bar{B} = \begin{bmatrix} 0 \\ 0 \\ T^3 \end{bmatrix}, \bar{F} = \begin{bmatrix} 0 \\ 0 \\ T^3 \end{bmatrix}, \bar{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \bar{G} = g_0$$
$$\alpha_1 = Ta_3 - 3,\ \ \alpha_2 = T^2 a_2 - 2Ta_3 + 3,\ \ \alpha_3 = T^3 a_1 - T^2 a_2 + Ta_3 - 1$$

Now that the system has been transformed into the discrete-time domain, the procedure for developing the system and designing the controller outlined in Chapter 2 can be applied. With the sampling time T introduced into the discrete-time system, other considerations to guarantee stability will be discussed.

4.2: Jury's Stability Test

Now that the sampling time T has been introduced into the system through the discretization process, it is important to investigate the effect of T on the stability of the closed-loop system. The application of Jury's Stability Test will provide a set of necessary stability conditions on the discrete-time system [11]. The conditions determine the location of the roots of a polynomial in the z-domain, with respect to the unit circle. In this case, Jury's Stability Test will be used to determine whether or not the eigenvalues of the characteristic equation are stable, or within the unit circle. It is beneficial to use this stability test after discretizing the originally continuous-time system described in the previous section to ensure that the sampling time T will not jeopardize the stability of the system. Applying Jury's Stability Test conditions to the discretized system will result in conditions on the sampling time T that will ensure the system's stability.

To begin, the discrete-time system's characteristic equation must be determined. This can be done by using the following equation, where A_{CL} represents the closed-loop state coefficient matrix [10].

$$\left|\lambda I_n - A_{CL}\right| = 0 \tag{4.2.1}$$

The resulting characteristic equation is a polynomial of degree n, where the roots of the polynomial are the eigenvalues of the system.

$$F(\lambda) = \beta_n \lambda^n + \beta_{n-1}\lambda^{n-1} + \ldots + \beta_2 \lambda^2 + \beta_1 \lambda + \beta_0 \tag{4.2.2}$$

In (4.2.2) β_1, β_2,…β_n represent the characteristic equation coefficients for the n^{th} order system. Jury's Stability test will be applied to the first, second, and third-order systems discretized in the previous section, 4.1.

4.2.1: First-order Systems

The characteristic equation for the first-order system was determined according to (4.2.1).

$$F(\lambda) = \left(T^2 + (\phi T + \gamma)^2\right)\lambda - \gamma(1 + Ta)(\phi T + \gamma) \tag{4.2.3}$$

According to Jury's Stability Test, the following conditions must be true for a first-order polynomial in order for the eigenvalues to be within the unit circle:

$$\begin{gathered} F(1) > 0 \\ F(-1) < 0 \\ |\beta_0| < \beta_1 \end{gathered} \tag{4.2.4}$$

Solving F(λ) for the value of one as in the first condition yields the following condition.

$$\left(T^2 + (\phi T + \gamma)^2\right) - \gamma(1 + Ta)(\phi T + \gamma) > 0 \tag{4.2.5}$$

Manipulations result in the following condition on the sampling time *T* and *a,* the original continuous-time system coefficient from (4.1.1).

$$T < -\frac{1}{a} \tag{4.2.6}$$

Considering the second condition from Jury's Stability Test, substituting the characteristic equation yields the following condition.

$$-\left(T^2 + (\phi T + \gamma)^2\right) - \gamma(1 + Ta)(\phi T + \gamma) < 0 \tag{4.2.7}$$

It can be seen in this condition that it will always be true, as the left side of the inequality will always be negative and less than zero. The last condition to satisfy Jury's Stability Test has been applied to the first-order system and is shown below.

$$\left|\gamma(1+Ta)(\phi T+\gamma)\right| < T^2 + (\phi T+\gamma)^2 \tag{4.2.8}$$

Rewriting the inequality results once again in a condition on *a,* the continuous-time system coefficient in terms of the sampling time *T*, and pseudo-output weighting coefficients.

$$a < \frac{T+\phi(T\phi+\gamma)}{\gamma(T\phi+\gamma)} \tag{4.2.9}$$

By satisfying the resultant coefficients, the first order system will be guaranteed to have eigenvalues within the unit circle.

4.2.2: Second-order Systems

The resulting characteristic equation for the second-order system previously discretized is shown below, where *T* is the sampling time, ϕ and γ are the pseudo-output coefficients, and α represents the coefficients of the discrete-time, canonical system.

$$F(\lambda) = (T^4+\gamma^2)\lambda^2 + (\gamma^2\alpha_2 + T^2\gamma\phi)\lambda + \alpha_1\gamma^2 \tag{4.2}$$

$$\alpha_1 = 2 - Ta_2, \;\; \alpha_2 = -T^2 a_1 + Ta_2 - 1$$

The stability conditions for a second-order system according to Jury's Stability Test are shown below, where β_0 and β_2 are coefficients from the characteristic equation.

$$F(1) > 0 \tag{4.2.11}$$

$$F(-1) > 0 \tag{4.2.12}$$

$$|\beta_0| < \beta_2 \tag{4.2.13}$$

Substituting this system's characteristic equation from (4.2.10) into the first condition (4.2.11) results in the equation below.

$$(T^4 + \gamma^2) + (\gamma^2 \alpha_2 + T^2 \gamma \phi) + \alpha_1 \gamma^2 > 0 \tag{4.2.14}$$

For this condition to be satisfied, the sampling time T should be chosen such that it is greater than zero, but much less than one.

$$0 < T << 1 \tag{4.2.15}$$

The second condition in Jury's stability test for the second-order system is that the characteristic equation solved for λ=-1 should be greater than zero (4.2.12).

$$(T^4 + \gamma^2) - (\gamma^2 \alpha_2 + T^2 \gamma \phi) + \alpha_1 \gamma^2 > 0 \tag{4.2.16}$$

In order for this condition to be satisfied, phi and gamma must be greater than zero.

$$\gamma > 0, \;\; \phi > 0 \tag{4.2.17}$$

The third and final condition is that the magnitude of the third characteristic equation coefficient should be less than the first characteristic equation coefficient (4.2.13).

$$|\alpha_1 \gamma^2| < (T^4 + \gamma^2) \tag{4.2.18}$$

Substituting the previously defined expression for α_1, the condition now is in terms of the original continuous-time system parameters a_1 and a_2. In order to fulfill this condition, a condition on the continuous-time system parameter a_2 and the sampling time T resulted.

$$T > \frac{1}{a_2} \tag{4.2.19}$$

By choosing parameters such that the three resulting conditions are satisfied (4.2.11-4.2.13), it will be guaranteed that the eigenvalues of the system will be within the unit circle, and will therefore be stable.

4.2.3: Third-order Systems

Now that conditions on the first and second-order systems have been developed from Jury's Stability test, they are applied to the third-order system. The characteristic equation for the previously developed discrete-time third-order system is shown below.

$$F(\lambda) = (T^6 + \gamma^2)\lambda^3 + (\alpha_3\gamma^2)\lambda^2 + (\alpha_2\gamma^2 + T^3\gamma\phi)\lambda + (\alpha_1\gamma^2) \qquad (4.2.20)$$

According to Jury's Stability Test for a third-order system, the following three conditions must be satisfied to ensure that the eigenvalues will be within the unit circle.

$$F(1) > 0 \qquad (4.2.21)$$

$$F(-1) < 0 \qquad (4.2.22)$$

$$|\beta_0| < \beta_3 \qquad (4.2.23)$$

Once again, a_0 and a_3 shown in the third condition (4.2.23) represent the first and last coefficients of the characteristic equation shown above in (4.2.20). Substituting the characteristic equation for λ=1 into the first condition yields the following condition, where once again *T* is the sampling time, ϕ and γ are the pseudo-output coefficients, and α represents the coefficients of the discrete-time, canonical system.

$$(T^6 + \gamma^2) + (\alpha_3\gamma^2) + (\alpha_2\gamma^2 + T^3\gamma\phi) + (\alpha_1\gamma^2) > 0 \qquad (4.2.24)$$

$$\alpha_1 = Ta_3 - 3,\ \ \alpha_2 = T^2a_2 - 2Ta_3 + 3,\ \ \alpha_3 = T^3a_1 - T^2a_2 + Ta_3 - 1$$

Substituting the corresponding expressions for α_1, α_2, and α_3 and manipulating the inequality, the following condition on a_1, the continuous-time canonical system coefficient, results.

$$a_1 > \frac{-(T+\gamma\phi)}{\gamma^2} \tag{4.2.25}$$

As long as a_1 is positive, the condition will always be satisfied. Considering now the second condition in Jury's Stability Test for third-order systems, F(λ) is solved for λ=-1.

$$-(T^6+\gamma^2)+(\alpha_3\gamma^2)-(\alpha_2\gamma^2+T^3\gamma\phi)+(\alpha_1\gamma^2)<0 \tag{4.2.26}$$

Once again substituting the expressions for α_1, α_2, and α_3 result in an inequality in terms of the original continuous-time canonical system coefficients a_1, a_2, and a_3. Manipulating the inequality results in the following condition on the system parameters a_1, a_2, and a_3.

$$a_2^{\ 2} > 4a_1a_3 \tag{4.2.27}$$

The final condition in Jury's Stability Test for third-order systems has been applied to the system's corresponding characteristic equation.

$$\left|\alpha_1\gamma^2\right| < T^6+\gamma^2 \tag{4.2.28}$$

Substituting the expression for α_1 results in a condition on the sampling time T and γ, shown below.

$$4T^2\left(\frac{T^6}{\gamma^2}+1\right)^2 > 0 \tag{4.2.29}$$

Inherently, this condition will always be true for values of γ greater than zero, as the sampling time T will always be positive.

By applying the resulting conditions for the first, second, and third-order continuous-time systems when beginning the design process, it can be guaranteed that the eigenvalues of the system will be within the unit circle. Such an analysis could be very beneficial to ensure that particular selected parameter values and sampling time will not jeopardize the stability of the system.

4.3: Controller Comparisons

To further the considerations for the implementation of the discrete-time, disturbance accommodation controller proposed in this work, a comparative analysis between with the deadbeat disturbance accommodation controller designed in [10] will be performed. When investigating the properties and characteristics of a particular controller, comparative analyses are beneficial to understand of the controllers' capabilities. The comparative analysis performed in this section will compare the discrete-time DAC design proposed in this work with the previously proposed deadbeat DAC through the use of a second-order example system.

The deadbeat disturbance accommodation controller proposed in [11] uses a reduced-order deadbeat observer-based design to provide estimates of the system states and disturbance. A deadbeat based control design will guarantee the fastest possible response of the system, placing the eigenvalues of the system at the origin. Although this deadbeat control design has a time-optimal response, the design does not take into account the lack of control term present in many system models. The system model and disturbance model used in this control design is shown below.

$$x(k+1) = Ax(k) + Bu(k) + Fw(k) \tag{4.3.1}$$

$$y(k) = Cx(k) + Du(k) + Gw(k) \tag{4.3.2}$$

$$w(k+1) = Ew(k) + \sigma_k \tag{4.3.3}$$

For the comparative example and analysis between controllers, the following coefficient matrices will be used.

$$A = \begin{bmatrix} 0 & 1 \\ -0.6329 & 1.807 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ F = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix},\ D = 0,\ G = 0.2,\ E = 1 \tag{4.3.4}$$

This controller design created a composite system composed of the state estimate update equation $\hat{x}(k+1)$ and output equation y(k), whereas the discrete-time DAC design proposed in this work used the concept of a "pseudo-output" instead.

$$\begin{bmatrix} \hat{x}(k+1) \\ y(k) \end{bmatrix} = A_c \begin{bmatrix} \hat{x}(k) \\ y(k-1) \end{bmatrix} + B_c u(k) + F_c w(k)$$
$$A_c = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix},\ B_c = \begin{bmatrix} B \\ D \end{bmatrix},\ F_c = \begin{bmatrix} F \\ G \end{bmatrix} \tag{4.3.5}$$

The controller gain K_c and disturbance gain K_d were designed using similar techniques to the control and disturbance gains designed in the second chapter of this work, and are shown below.

$$K_c = -(B_c{}^T B_c)^{-1} B_c{}^T A_c$$
$$K_d = -(B_c{}^T B_c)^{-1} B_c{}^T F_c \tag{4.3.6}$$

With the control and disturbance gains obtained, the closed-loop system is determined and is defined according to the equation below.

$$\begin{bmatrix} \hat{x}(k+1) \\ y(k) \end{bmatrix} = \left(A_c + B_c K_c\right) \begin{bmatrix} \hat{x}(k) \\ y(k-1) \end{bmatrix} + \left(F_c + B_c K_d\right) w(k) \tag{4.3.7}$$

The simulation of this controller applied to the second-order example system yield the output and control input trajectories shown in Figure 4.1 and 4.2 below. Although the response is indeed deadbeat, the magnitude of the control input response is very high, proving impractical for implementation.

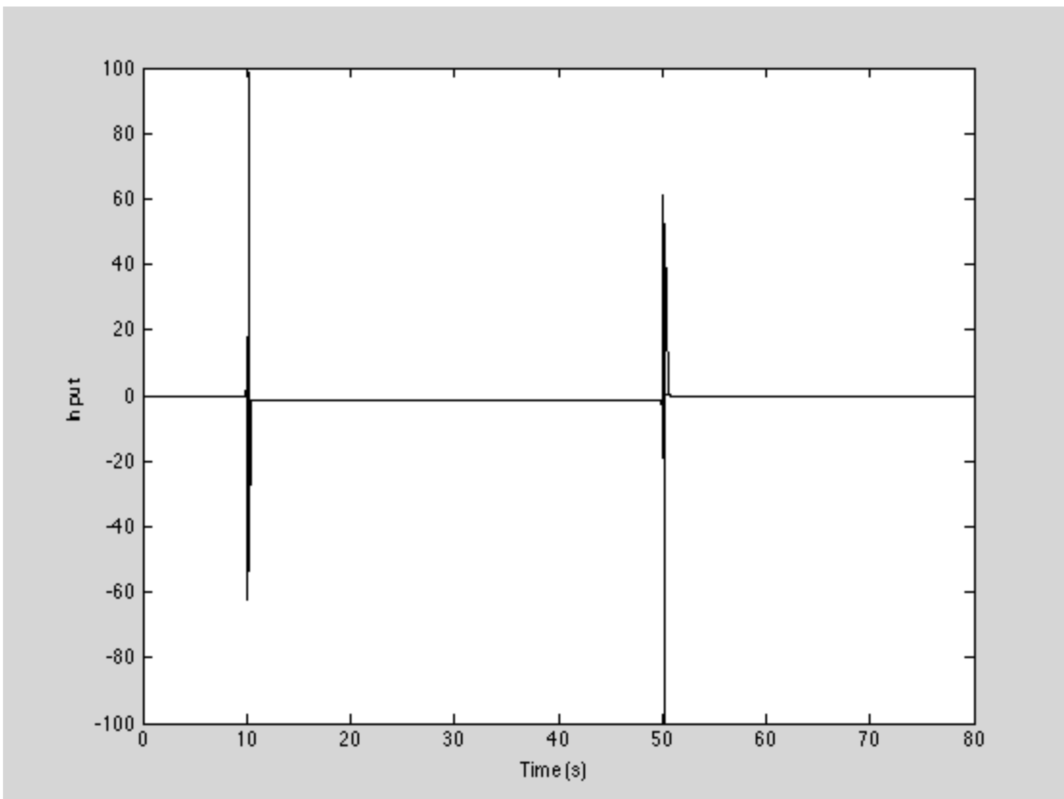

Figure 4. 1: Deadbeat DAC Input

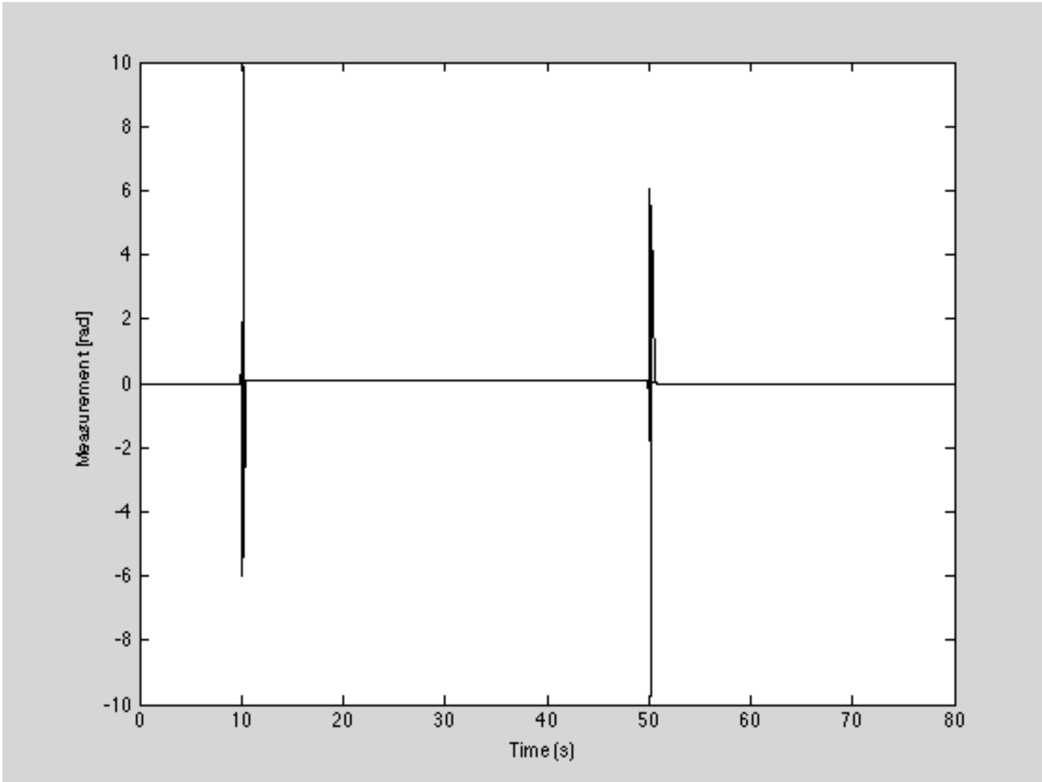

Figure 4. 2: Deadbeat DAC Output

Now that the previous control design has been applied to the system, the comparative analysis will turn back to the control design proposed in this work. Once again, the discrete-time disturbance accommodation controller proposed here introduced the concept of a "pseudo-output", z(k). The purpose of the pseudo-output is to introduce a control term in the output equation of systems where it would normally be missing. Although this controller is not deadbeat, a moderately fast response is still achieved, while minimizing the control input to result in more practical implementation.

Turning to the controller proposed in this work and its analysis methods discussed in Chapter 3, the second-order comparative example applied to the deadbeat disturbance accommodation controller in (4.3.4) is considered. Upon performing the stability analysis investigating the closed-loop eigenvalues as functions of the pseudo-output weighting parameters and disturbance accommodation analysis considering the disturbance Grammian versus the same pseudo-output weighting parameters, a feasible range of parameter values is found. This region of acceptable ϕ and γ values is shown in Figure 4.3.

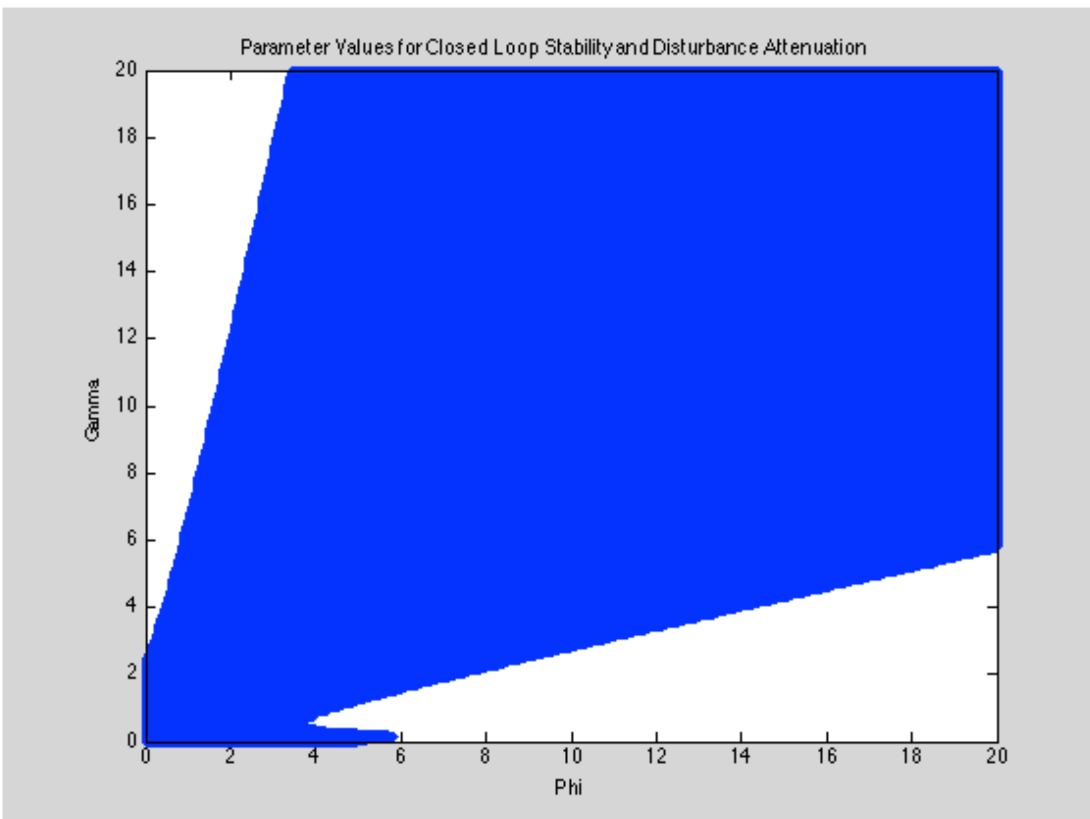

Figure 4. 3: Region of Closed-loop Stability and Disturbance Attenuation

Now that an acceptable range of values has been found, it is beneficial to determine the cost associated with selecting particular values of ϕ and γ. The cost analysis is performed by minimizing the cost function in (2.21) and analyzing it versus ϕ and γ.

The figure below shows the results of the cost function analysis for the second-order example, where the cooler colors represent the regions of lowest cost.

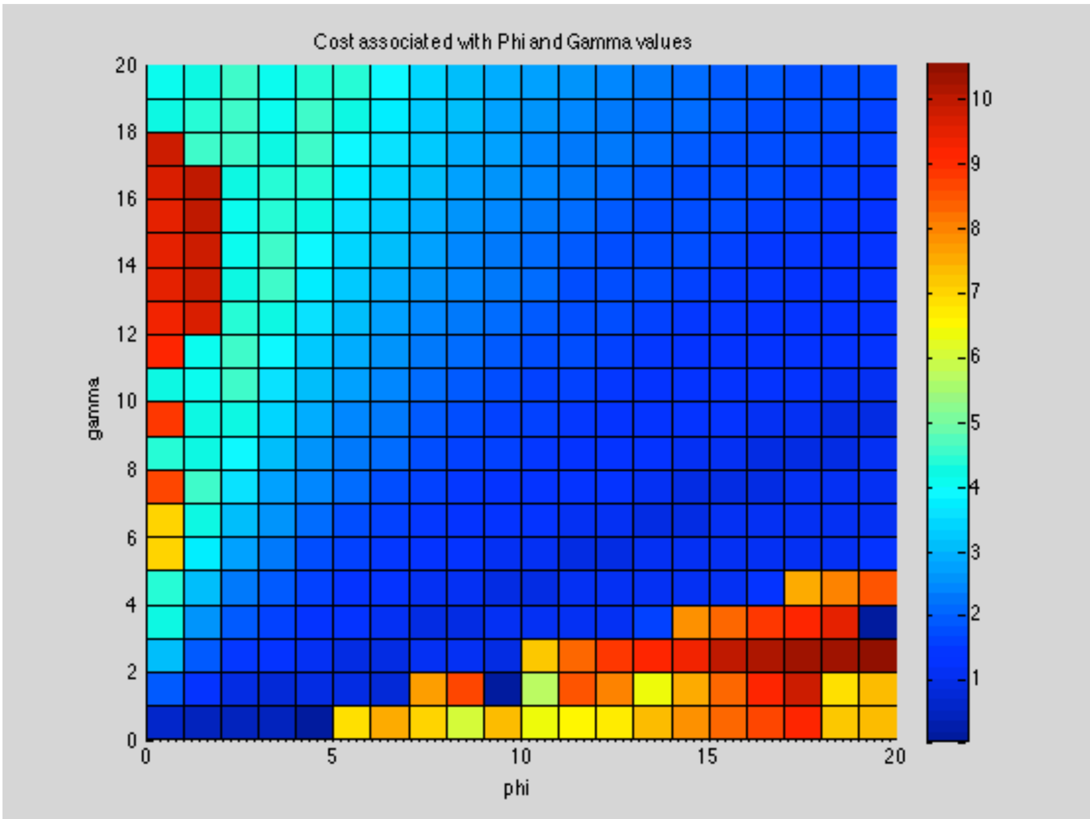

Figure 4. 4: Cost Analysis

Taking the cost, stability, and disturbance accommodation analyses into consideration, the second-order comparative example was simulated for ϕ and γ equal to one, values in the lowest cost region of the figures. The corresponding output and control input trajectories are shown in the figures below.

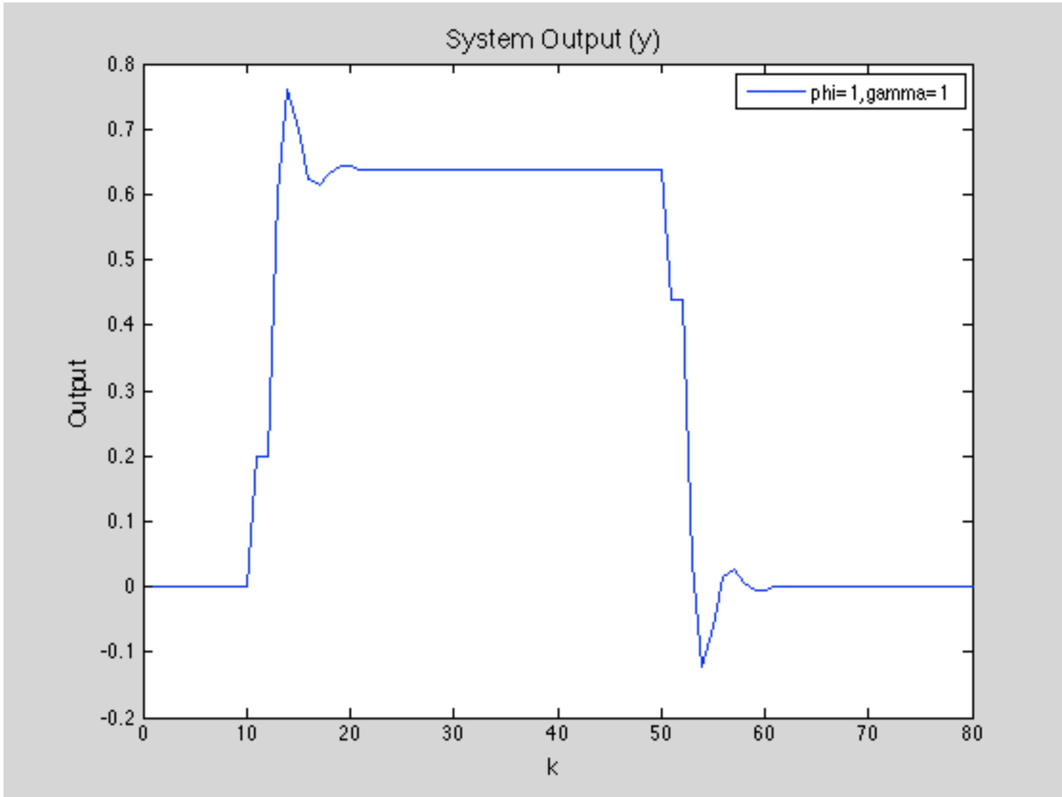

Figure 4. 5: DAC Output

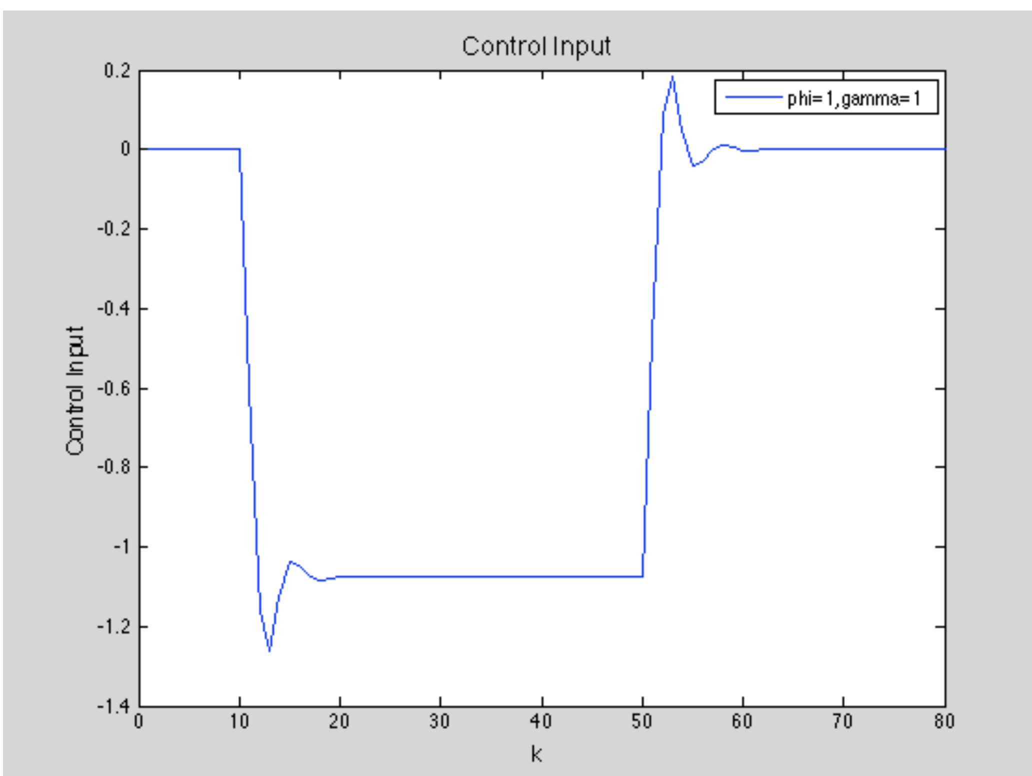

Figure 4. 6: DAC Input

Taking particular note to the magnitude of the control input response shows that the discrete-time, DAC proposed in this work does indeed meet the objective of minimizing the control input for more feasible implementation on the previous deadbeat control design. Although the response is no longer time-optimal, certain applications may deem this particular design more feasible due to the minimized control input while still minimizing the effects of the disturbance in the output.

Chapter 5: Conclusion

5.1: Contribution

Although there have been previous controller designs in the area of disturbance accommodation control theory for linear and non-linear systems, there was a need for a controller design to compensate the lack of control term in the output equation of many systems. This work considered systems in discrete-time canonical form, as well as considerations for the implementation of such systems. The control techniques proposed in this work introduced the concept of a "pseudo-output" composed of the previous input term and current output term, with corresponding weighting parameters ϕ and γ. The pseudo-output is designed to add an input term to the output equation where it would normally be missing. By applying this technique and taking the lack of input term in the output equation into consideration, more favorable results were achieved.

Three main controller objectives were considered in the design and analysis of this discrete-time disturbance accommodation controller. Firstly, it was crucial to the performance of the system that the closed-loop system be stable. Therefore, a stability analysis was performed to analyze the closed-loop eigenvalues as a function of the pseudo-output weighting parameters. Secondly, the controller dampened the effects of the disturbance of known waveform applied to the system. To quantify and analyze the controller's disturbance attenuation capabilities, the concept of a disturbance grammian was introduced and analyzed versus the pseudo-output weighting parameters. Considering both of these important analyses, an acceptable range of parameter values for the pseudo-output weighting parameters achieving both stability and disturbance accommodation was found. The third controller objective in this work was to minimize the magnitude of the control input was done by minimizing the norm of the control gains. A cost function was defined minimizing the input and output, based from a regulator-based control design. Following the analysis of the stability, disturbance accommodation and control input minimization, the state trajectories of the systems were simulated for values of the pseudo-output weighting parameters within the allowable range of values found in the analysis. In this manner, the effect of particular parameter values could be seen, and the analysis validated.

To demonstrate the controller and the controller objectives, first, second and third-order systems were considered with varying degrees of stability. By considering these types of systems, the performance of the controller under various circumstances was investigated.

5.2: Summary of Results

Application of the disturbance accommodation controller to the discrete-time systems in each case study and corresponding analyses resulted in general trends, depending on the selection of the pseudo-output weighting parameters. Upon finding an acceptable range of phi and gamma parameter values to guarantee stability and disturbance accommodation, similar points were chosen to simulate the output response, input response, and state trajectories. By considering various values of phi and gamma for each system, trends were observed for not only each system and case study, but for all of the systems in general.

For each first, second, or third-order system of varied open-loop stability, the stability and disturbance accommodation analysis yielded an acceptable region of phi and gamma to achieve those objectives. It was observed for the second and third-order systems, that the region of phi and gamma values decreased in size as the open-loop system decreased in stability. Although this was not observed in the first-order case, it was to be expected as every value of phi and gamma was able to result in closed-loop stability and disturbance accommodation. Upon analysis of the cost function, it was also visible that a higher cost of implementing particular phi and gamma values was associated with each system of first, second, and third-order. Keeping in mind that the cost function represents a minimization of the control input and output over time, it is understandable that a higher input magnitude is required to achieve the same results in an unstable system as for a stable or less unstable system.

Upon obtaining an acceptable range of phi and gamma values for each case study, particular values could be chosen to investigate the output response, input response, and state trajectories of each system. It was observed in the results for each system and case study that as the value of phi increased, the magnitude of the output response decreased while the magnitude of the input response increased. Not only was there a trend in the magnitude of the responses, but also in the magnitude and settling time of the transient response. As phi increased, the transient response also increased. This is an effect that could greatly affect the application of this controller to particular sensitive applications.

Considering the other pseudo-output weighting parameter, gamma, trends were also visible as the value of gamma was increased. Increasing gamma resulted in a higher output response magnitude, while the magnitude of the input response decreased, conversely to the effect of phi on the system. Furthermore, the magnitude of the transient response as well as the associated settling time decreased as the value of gamma was increased.

Another objective in the design of this controller was to minimize the input magnitude, compared to a deadbeat DAC design whose magnitude of the control input proved impractical for implementation [11]. The results of a comparative analysis between the performance of the controller proposed in this work and the previous designed controller verified that the magnitude of the control input could indeed be minimized to result in a more feasible implementation.

5.3: Future Work

Although the disturbance accommodation control design proposed in this work achieved the considered objectives, other extensions of this design could be investigated in the future. This design included a step-type disturbance applied to each of the systems in the design and case studies but other known waveform type disturbances such as ramp or sinusoidal-type disturbance signals could be used in combination with or in place of a step. In this work, first, second and third order systems were considered. The results from the analysis of the systems in this work could be extended to yield results for any n^{th} order system. Lastly, for this work, it was assumed that the state variables were available for use. The same investigations proposed in this work could be performed using an estimator and the differences between the methods investigated.

BIBLIOGRAPHY

[1] C.D. Johnson, G.A. Parker, "Improved Speed Regulation and Mitigation of Drive Train Torsion Fatigue in Flexible Wind Turbines, Using Disturbance Utilization Control: Parts One and Two", 41st Southeastern Symposium on System Theory, 2009. SSST 2009, pp.171-183, 15-17 March 2009.

[2] C.D. Johnson, "Disturbance-Accommodation Control: An Overview of the Subject", Journal of Interdisciplinary Modeling and Simulation, 3(1), 1980, pp.1-29.

[3] C.D. Johnson, "Accommodation of External Disturbances in Linear Regulator and Servomechanism Problems", IEEE Transactions on Automatic Control, 16(6), 1971, pp. 635-644.

[4] A. Azemi, E. Yaz, "Dynamic Disturbance Minimization Control for Discrete Nonlinear Stochastic Systems", Optimal Control Applications and Methods, 42(7), 1994, pp. 2480-2489.

[5] E. Yaz, "Accommodation for Disturbances having Nonlinear Models Acting on Nonlinear Systems", IEEE Transactions on Circuit and Systems, 33(3), 1986, pp. 342-343.

[6] I. M. Tshiofwe, M. Alotaibi, J. Amirazodi, E.E. Yaz, "An LMI-Based Disturbance Reduction Control for Systems with Multiple State and Input Delays", Proceedings of the Conference on Decision and Control, Orlando, FL, 2001, pp. 1444-1445.

[7] I.M. Tshiofwe, M. Al-Harthi, M. Alotaibi, J. Amirazodi, E.E. Yaz, "A Reduced-Order Compensator for Suppression of Process and Measurement Disturbances", Proceedings of the American Control Conference, Denver, CO, 2003, pp. 3886-3891.

[8] H. Kim, Y. Kim, "Recursive Control Design for Low Frequency Disturbance Accommodation", IEEE Transactions on Aerospace and Electronic Systems, 36(4), 2000, pp. 1330-1335.

[9] J.L. Riffer, "Time-Optimal Control of Discrete-Time Systems with Known Waveform Disturbances", Master's Thesis, Marquette University (August 2009).

[10] R.J. Vaccaro <u>"Digital Control: A State-Space Approach"</u> (NY, McGraw-Hill, 1995).

[11] J.L. Riffer, K.H. Barhouse, S.C. Schneider, E.E. Yaz, "Constrained Time-Optimal Control of Discrete-Time Systems with Known Waveform Disturbances",

Proceedings of IASTED International Conference on Circuits and Systems. Maui, HI, pp.218-224 (2010).

[12] B.C. Kuo, "Digital Control Systems", 2nd Ed. Oxford University Press, New York, NY, 1992.

Appendix: MATLAB Code

MATLAB was used to simulate each of the systems considered in this work. For each system of first, second, and third order, the same simulation techniques were used. Therefore, in this appendix, the code used to simulate the third-order systems is shown below in A1, A2 and A3.

A1: Closed-loop Stability and Disturbance Accommodation Analysis Code:

```
% THIRD ORDER SYSTEM SIMULATIONS
% DISTURBANCE GRAMMIAN, STABILITY, AND PHI/GAMMA REGION PLOTS

close all
clear all
clc
%DEFINITIONS
kmax=500;
z(kmax,2)=zeros;       % NOT pseudo output...for determining stable
DB=zeros(kmax,kmax);   % phi and gamma values disturbance grammian
gamma=zeros(1,kmax);
phi=zeros(1,kmax);
AA=zeros(4,4);
x(4,kmax)=zeros;
u(1,kmax)=zeros;
w(1,kmax)=zeros;
wk(1,kmax)=ones(size(kmax));
T=0.4;
Qc=eye(4);Qd=eye(4); Rc=0; Rd=0;
% DISCRETE-TIME SYSTEM IN CANON FORM
%alpha1=-0.04; alpha2=0.53; alpha3=-1.4;  % stable
%alpha1=-0.055; alpha2=0.71; alpha3=-1.7; % slightly unstable
alpha1=-0.065; alpha2=0.83; alpha3=-1.9; % more unstable
Abar=[0 1 0;0 0 1;-alpha1 -alpha2 -alpha3];
Bbar=[0;0;1];
Cbar=[1 0 0];
Fbar=Bbar;  % F equal to B
Gbar=T^3;    % SMALL value for G
E=1;
Wcbar=ctrb(Abar,Bbar);
eig_Wcbar=eig(Wcbar'*Wcbar);
% Gamma and Phi
gamma_min=0;
gamma_max=20;
delta_gamma=(gamma_max-gamma_min)/kmax;
gamma=gamma_min:delta_gamma:gamma_max;
phi_min=0;
phi_max=20;
delta_phi=(phi_max-phi_min)/kmax;
phi=phi_min:delta_phi:phi_max;
%Calculate Augmented closed loop system matrices
for i=1:kmax+1; % Gamma
```

```
    for j=1:kmax+1; %Phi
        %augmented sys
        Ap=[Abar [0;0;0];(phi(1,j)*Cbar*Abar) 0];
        Bp=[Bbar;(phi(1,j)*Cbar*Bbar)+gamma(1,i)];
        Fp=[Fbar;phi(1,j)*(Cbar*Fbar+Gbar*E)];
        % gains
        Kd=-inv(Bp'*Qd*Bp+Rd)*Bp'*Qd*Fp;
        Kc=-inv(Bp'*Qc*Bp+Rc)*Bp'*Qc*Ap;
        % closed loop
        A_cl=(Ap+Bp*Kc);
        F_cl=(Fp+Bp*Kd);
        eig_hold=eig(A_cl);     % store eigenvalues of closed loop
system matrix (A_cl)
        eigen1(i,j)=abs(eig_hold(1));
        eigen2(i,j)=abs(eig_hold(2));
        eigen3(i,j)=abs(eig_hold(3));
        eigen4(i,j)=abs(eig_hold(4));
        DB(i,j)=F_cl'*F_cl;      % magnitude of closed loop disturbance
attenuation
        maxEIG12(i,j)=max(eigen1(i,j),eigen2(i,j));
        maxEIG34(i,j)=max(eigen3(i,j),eigen4(i,j));
        maxEIG(i,j)=max(maxEIG12(i,j),maxEIG34(i,j));

    end
end
k=0; % reset index k to zero
%apply conditions to assure stability
for i=1:kmax+1;   %gamma
    for j=1:kmax+1;   %phi
        if((eigen1(i,j)<1)&& (eigen2(i,j)<1) && (eigen3(i,j)<1)
&&(eigen4(i,j)<1)...
            && (DB(i,j)<1) )
            k=k+1;
            z(k,1)=phi(1,j);
            z(k,2)=gamma(1,i);

        end
    end
end

figure(1)
plot(z(:,1),z(:,2),'g','Marker','.','LineStyle','none'),hold on
xlabel('Phi','fontsize',12)
ylabel('Gamma','fontsize',12)
title('Parameter Values for Closed Loop Stability and Disturbance
Attenuation')
axis([0 20 0 20])

figure(2)
mesh(phi, gamma,DB)
xlabel('Phi')
ylabel('Gamma')
zlabel('Attenuation')
title('Disturbance Attenuation as a Function of Control Parameters')
grid on

figure(3)
```

```
mesh(phi,gamma,maxEIG)
xlabel('Phi')
ylabel('Gamma')
zlabel('Max Eigenvalue')
title('Maximum Eigenvalues as a function of Control Parameters')
grid on
```

A2: Cost Analysis Code:

```
% Third-order systems- Cost Analysis
% Katrina Barhouse- Thesis simulations
close all
clear all
clc
%DEFINITIONS
kmax=500;
z(kmax,2)=zeros;     % NOT pseudo output...
DB=zeros(kmax,kmax);   % disturbance grammian
gamma=zeros(1,kmax);   %nmax or nmax+1???
phi=zeros(1,kmax);
AA=zeros(4,4);
x(4,kmax)=zeros;
u(1,kmax)=zeros;
w(1,kmax)=zeros;
wk(1,kmax)=ones(size(kmax));
T=0.4;
Qc=eye(4);Qd=eye(4); Rc=0; Rd=0;

%DISCRETE-TIME CANONICAL SYSTEM
%alpha1=-.04; alpha2=0.53; alpha3=-1.4  % stable sys
%alpha1=-0.055; alpha2=0.71; alpha3=-1.7; % slightly unstable
alpha1=-0.065; alpha2=0.83; alpha3=-1.9; %unstable

Abar=[0 1 0;0 0 1;-alpha1 -alpha2 -alpha3];
Bbar=[0;0;1];
Cbar=[1 0 0];
Fbar=Bbar;  % F equal to B
Gbar=T^3;    % SMALL value for G
E=1;
Wcbar=ctrb(Abar,Bbar);
eig_Wcbar=eig(Wcbar);
% generate step type disturbance w
sigma=zeros(1,kmax);
sigma(1,100)=1;
sigma(1,300)=-1;
% set up Gamma and Phi
gamma_min=1;
gamma_max=21;
gamma_step=gamma_min:1:gamma_max;
count_gamma=length(gamma_step);
phi_min=1;
phi_max=21;
phi_step=phi_min:1:phi_max;
count_phi=length(phi_step);

 for i=1:count_gamma
     for j=1:count_phi
```

```
         gamma(1,:)=gamma_step(1,i);
         phi(1,:)=phi_step(1,j);

        %Calculate Augmented closed loop system matrices
        %augmented sys
        Ap=[Abar [0;0;0];(phi(1,j)*Cbar*Abar) 0];
        Bp=[Bbar;(phi(1,j)*Cbar*Bbar)+gamma(1,i)];
        Fp=[Fbar;phi(1,j)*(Cbar*Fbar+Gbar*E)];
        Cp=[Cbar 0];
        Gp=Gbar;
        % gains
        Kd=-inv(Bp'*Qd*Bp+Rd)*Bp'*Qd*Fp;
        Kc=-inv(Bp'*Qc*Bp+Rc)*Bp'*Qc*Ap;
        % closed loop
        A_cl=(Ap+Bp*Kc);
        F_cl=(Fp+Bp*Kd);

        %Simulate state trajetories
        for k=1:kmax
            if(x(:,k)<10)
            w(1,k+1)=w(1,k)+sigma(1,k);
            x(:,k+1)=A_cl*x(:,k)+F_cl*w(:,k);
            u(:,k)=Kc*x(:,k)+Kd*w(:,k);
            y(:,k)=Cp*x(:,k)+Gp*w(:,k);
            else
                u(:,k)=0;
                y(:,k)=0;
            end
        end
% cost
        J=u.^2+y.^2;
        Jmax=max(J);
        Jfinal2(i,j)=Jmax;
        Cost(i,j)=log(Jfinal2(i,j));   % plot cost on logarithmic scale
        end
 end

[a,b] = meshgrid(0:1:20, 0:1:20);
surf(a,b,Cost);
xlabel('phi')
ylabel('gamma')
zlabel('Cost')
title('Cost associated with Phi and Gamma values')
```

A3: State Trajectory Response Code

```
% Third order system
% D=0,Simulation of state trajectories for specific phi and gamma
values
%close all
clear all
clc
%DEFINITIONS
%======================
phi=1;
gamma=4;
```

```
%=========================
kmax=500;
z(kmax,3)=zeros;
DB=zeros(kmax,kmax);   % disturbance grammian

AA=zeros(4,4);
x(4,kmax)=zeros;
u(1,kmax)=zeros;
y(1,kmax)=zeros;
w(1,kmax)=zeros;
wk(1,kmax)=ones(size(kmax));

T=0.4;
Qc=eye(4);Qd=eye(4); Rc=0; Rd=0;

%DISCRETE-TIME CANON SYSTEM
%alpha1=-.04; alpha2=0.53; alpha3=-1.4  % stable sys
%alpha1=-0.055; alpha2=0.71; alpha3=-1.7; % slightly unstable
alpha1=-0.065; alpha2=0.83; alpha3=-1.9; %unstable

Abar=[0 1 0;0 0 1;-alpha1 -alpha2 -alpha3];
Bbar=[0;0;1];
Cbar=[1 0 0];
Fbar=Bbar;  % F equal to B
Gbar=T^3;    % SMALL value for G
E=1;
Wcbar=ctrb(Abar,Bbar);
eig_Wcbar=eig(Wcbar);
% generate step type disturbance w
sigma=zeros(1,kmax);
sigma(1,100)=1;
sigma(1,300)=-1;
%Calculate Augmented closed loop system matrices
        %augmented sys
        Ap=[Abar [0;0;0];(phi*Cbar*Abar) 0];
        Bp=[Bbar;(phi*Cbar*Bbar)+gamma];
        Fp=[Fbar;phi*(Cbar*Fbar+Gbar*E)];
        Cp=[Cbar 0];
        Gp=Gbar;
        % gains
        Kd=-inv(Bp'*Qd*Bp+Rd)*Bp'*Qd*Fp;
        Kc=-inv(Bp'*Qc*Bp+Rc)*Bp'*Qc*Ap;
        % closed loop
        A_cl=(Ap+Bp*Kc);
        F_cl=(Fp+Bp*Kd);
        eig=eig(A_cl);     % store eigenvalues of closed loop system
matrix (A_cl)
        DB=F_cl'*F_cl;      % magnitude of closed loop disturbance
attenuation

%Simulate state trajetories
for k=1:kmax-1
    w(1,k+1)=w(1,k)+sigma(1,k);
    x(:,k+1)=A_cl*x(:,k)+F_cl*w(:,k);
    u(:,k)=Kc*x(:,k)+Kd*w(:,k);
    y(:,k)=Cp*x(:,k)+Gp*w(:,k);
end
```

```
% Calculate Cost of Phi/Gamma point
J=u.^2+y.^2;
Jmax=max(J);
Jfinal=Jmax;
Cost=log(Jfinal)


k=1:1:kmax;
lg=['phi=0.5,gamma=1',
    'phi=0.5,gamma=4',
    'phi=1,gamma=4  '];
% lg=['phi=0.5,gamma=1 ',
%     'phi=0.5,gamma=6 ',
%     'phi=2,gamma=6   '];
figure(1)
 plot(k,x(1,:),'g'), hold on
 title('State Trajectory x1','fontsize',14)
 xlabel('k','fontsize',12)
 ylabel('x1','fontsize',12)
legend(lg)
figure(2)
 plot(k,x(2,:),'g'), hold on
 title('State Trajectory x2','fontsize',14)
 xlabel('k','fontsize',12)
 ylabel('x2','fontsize',12)
 legend(lg)
figure(3)
 plot(k,x(3,:),'g'), hold on
 title('State Trajectory x3','fontsize',14)
 xlabel('k','fontsize',12)
 ylabel('x3','fontsize',12)
 legend(lg)
figure(4)
 plot(k,x(4,:),'g'), hold on
 title('State Trajectory (z)','fontsize',14)
 xlabel('k','fontsize',12)
 ylabel('x4','fontsize',12)
 legend(lg)
figure(5)
 plot(k,u(:,:),'g','LineWidth',1), hold on
 title('Control Input','fontsize',14)
 xlabel('k','fontsize',12)
 ylabel('Control Input','fontsize',12)
 legend(lg)
figure(6)
 plot(k,y(:,:),'g','LineWidth',1), hold on
 title('System Output (y)','fontsize',14)
 xlabel('k','fontsize',12)
 ylabel('Output','fontsize',12)
 legend(lg)
figure(7)
 plot(k,w(:,:),'LineWidth',3), hold on
 title('Step Disturbance','fontsize',14)
 axis([0 500 0 1.2])
 label('k','fontsize',12)
 ylabel('Magnitude','fontsize',12)
```